Rational Hausdorff divisors: A new approach to the approximate parametrization of curves

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ABSTRACT

In this paper, we introduce the notion of rational Hausdorff divisor, we analyze the dimension and irreducibility of its associated linear system of curves, and we prove that all irreducible real curves belonging to the linear system are rational and are at finite Hausdorff distance among them. As a consequence, we provide a projective linear subspace where all (irreducible) elements are solutions of the approximate parametrization problem for a given algebraic plane curve. Furthermore, we identify the linear system with a plane curve that is shown to be rational and we develop algorithms to parametrize it analyzing its fields of parametrization. Therefore, we present a generic answer to the approximate parametrization problem. In addition, we introduce the notion of Hausdorff curve, and we prove that every irreducible Hausdorff curve can always be parametrized with a generic rational parametrization having coefficients depending on as many parameters as the degree of the input curve.

1. Introduction

When applying computational mathematics in practical applications, even though one may be dealing with a problem that can be solved algorithmically, and even though one has good algorithms to approach the solution, it can happen, and often it is the case, that the problem has to be reformulated and analyzed from a different computational point of view. This is the case of the development of approximate algorithms. This paper frames in the research area of approximate algebraic geometry and commutative algebra and, more precisely, on the problem of the approximate parametrization.

1.1. Approximate algebraic geometry and commutative algebra

We start with a subsection devoted to introduce informally the idea of approximate algorithm and to comment on some achievements in approximate algebraic geometry and commutative algebra.

Let $\mathcal{E}$ be a mathematical entity appearing in the resolution of a practical problem (e.g. $\mathcal{E}$ is a real polynomial) that is known, because of the nature of the treated applied problem, to satisfy certain property $\mathcal{P}$ (e.g. being reducible over $\mathbb{Q}$) that implies the existence of certain associated objects $\mathcal{E}_1, \ldots, \mathcal{E}_n$ (e.g. the irreducible factors over $\mathbb{Q}$ of the polynomial), and let the goal of the problem be to compute $\mathcal{E}_1, \ldots, \mathcal{E}_n$. However, often in practical applications, we receive a perturbation

$\mathcal{E}'$ of $\mathcal{E}$ instead of $\mathcal{E}$, where the property $\mathcal{P}$ does not hold anymore neither the associated objects $\mathcal{E}_i$ exist; for instance, a perturbation of a $\mathbb{Q}$-reducible polynomial will be, in general, $\mathbb{Q}$-irreducible and, therefore, the application of the existing polynomial factorization algorithms will just not solve our problem. One may try to recover the original unperturbed entity $\mathcal{E}$. Since, this is essentially impossible, a more realistic version of the problem is to determine a new object $\mathcal{E}''$ near $\mathcal{E}'$, satisfying $\mathcal{P}$, as well as computing the associated objects $\mathcal{E}_i''$ to $\mathcal{E}''$. An algorithm solving a problem of the above type is called an approximate algorithm; a solution of the illustrating example on polynomial factorization is given e.g. in [1].

One may distinguish two main phases when dealing with this type of problems. On one hand, the development of a theoretical reasoning that yields an algorithm and, on the other, providing an analysis of the distance between input and output in terms of a given tolerance. The distance used depends on the particular treated problem. For instance, in the factorization problem, if the input is $f \in \mathbb{C}[x_1, \ldots, x_n]$, and the output is $g \in \mathbb{C}[x_1, \ldots, x_n]$, one requires that $\|f - g\|_\infty$ is smaller than the tolerance. Algebraic varieties (that is, sets of points whose coordinates are zeros of some polynomials), as for instance algebraic curves and algebraic surfaces, are the main objects in algebraic geometry. Therefore, in approximate algebraic geometry (which is our case), in general, since one usually deals with sets, the Hausdorff distance is used. Indeed, the Hausdorff distance has proven to be an appropriate tool for measuring the resemblance between two geometric objects, becoming in consequence a widely used tool in fields as computer aided design, pattern matching and pattern recognition (see e.g. [2,3]); at the end of Section 1.2 we recall the definition of Hausdorff distance.

One can find in the literature papers treating this type of problems (say related to algebraic geometry and commutative algebra) with the same, or similar, strategies; see [4] for a general overview on approximate commutative algebra. Approximate algorithms for computing polynomial gcds can be found in [5–7], the polynomial factorization problem is addressed in [8,9,1,10]. For algebraic varieties there also exist approximate solutions: see [11–13] for the implicitization problem, see [14–16] for the study and analysis of singularities, and for the parametrization problem see [17–26]; approximate parametrization is the central topic of this paper and it will be commented in detail in the next subsections of this introduction.

1.2. Approximate parametrization problem

The approximate parametrization problem can be stated as follows (we state it for plane curves, but it can be similarly stated for space curves, see [25], and for surfaces, see [24]).

> Approximate parametrization problem for plane curves: Given the implicit equation of a non-rational real plane curve $\mathcal{C}$ and a tolerance $\epsilon > 0$, decide whether there exists a rational real plane curve $\overline{\mathcal{C}}$ at finite small distance (i.e. small related to the tolerance ϵ) to the input curve $\mathcal{C}$ and, in the affirmative case, compute a real rational parametrization of $\overline{\mathcal{C}}$.

The problem, as stated above, requires a global answer, that is, an algebraic curve $\overline{\mathcal{C}}$ to play the role of $\mathcal{C}$. However, not always such an answer exists. For instance, if one is interested in getting $\overline{\mathcal{C}}$ with the same topological graph as $\mathcal{C}$, the expected answer is that there will be no global solution; note that the genus of $\overline{\mathcal{C}}$ and $\mathcal{C}$ are different and the genus measures the difference between the maximum number of singularities and the actual one, counted properly, and this clearly affects the graph. So, the problem is often reformulated such that the solution is given in terms of piecewise rational curves; see for instance [18–21,26]. We, in our research (see [22–25]), are interested in the global answer, and hence, in the problem as it is stated above.

In our situation, the input curve $\mathcal{C}$ is supposed to have suffered a perturbation, in the sense that the coefficients of its defining polynomials have been perturbed. To treat the problem, even though the coefficients of our input polynomial have suffered a perturbation, and hence they are not the correct expected ones, we consider them as exact coefficients of our input and we work symbolically with them. For instance, let us consider the curve $\mathcal{C}$ defined by the implicit equation

$$x^4 + 2y^4 + \frac{1001}{1000}x^3 + 3x^2y - y^2x - 3y^3 + \frac{1}{100000}y^2 - \frac{1}{1000}x - \frac{1}{1000}y - \frac{1}{1000} = 0$$

and the curve $\overline{\mathcal{C}}$ defined by the implicit equation

$$x^4 + 2y^4 + \frac{1001}{1000}x^3 + 3x^2y - y^2x - 3y^3 = 0.$$

The curve $\mathcal{C}$ cannot be parametrized by means of rational functions, indeed its genus is 3, while $\overline{\mathcal{C}}$ can be parametrized e.g. by

$$\left(\frac{3000\,t^3 + 1000\,t^2 - 3000\,t - 1001}{1000(2\,t^4 + 1)}, \frac{t\left(3000\,t^3 + 1000\,t^2 - 3000\,t - 1001\right)}{1000(2\,t^4 + 1)}\right),$$

and one can see in Fig. 1 that both curves are close to each other.

But, above, what does it mean that $\mathcal{C}$ and $\overline{\mathcal{C}}$ are close? As mentioned before, we require that the Hausdorff distance between $\mathcal{C}$, $\overline{\mathcal{C}}$ is small related to a given tolerance ϵ. A main difficulty when working with the Hausdorff distance is that, if not both sets are bounded, the distance between them can be infinity. Most of the papers deal with bounded real algebraic

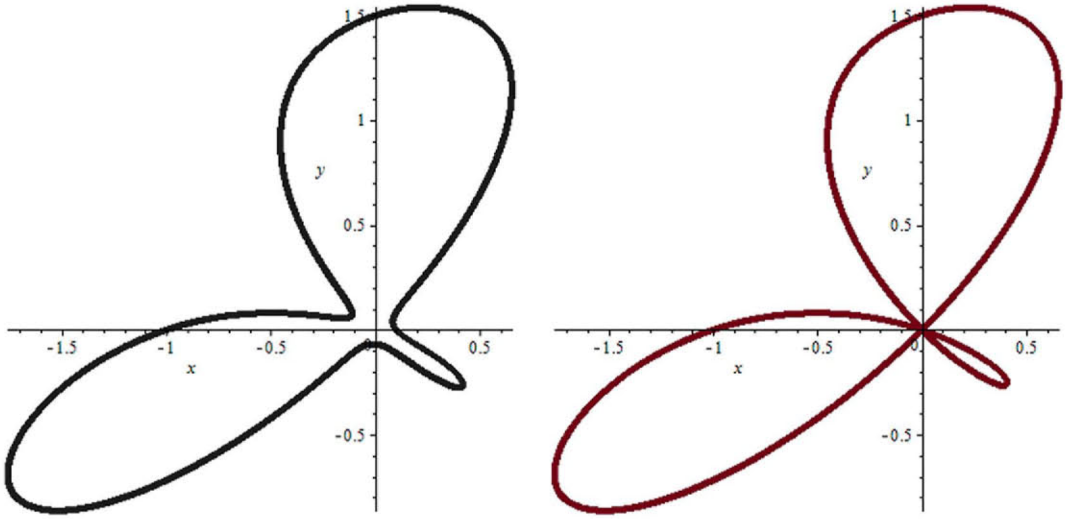

Fig. 1. Left: input curve $\mathcal{C}$. Right: output curve $\overline{\mathcal{C}}$.

curves or with parts of the curves framed into a box, and do not face the unbounded case. We, in our previous papers [22,25], do not restrict to the bounded case and we provide algorithms to derive *one* solution for the approximate parametrization problem. The aim of this paper is to describe the set of all curves $\overline{\mathcal{C}}$ that are solutions of the approximate parametrization problem for a given curve $\mathcal{C}$. If this set in known, one can approach the problem of choosing the best answer for the particular application from where the input curve comes from.

Hausdorff distance. We briefly recall the notion of Hausdorff distance; for further details we refer to [27]. In a metric space (X, d), for $\emptyset \neq B \subset X$ and $a \in X$ we define $\mathrm{d}(a, B) = \inf_{b\in B}\{\mathrm{d}(a, b)\}$. Moreover, for nonempty $A, B \subset X$ we define

$$\mathrm{H}_{\mathrm{d}}(A, B) = \max\left\{\sup_{a\in A}\{\mathrm{d}(a, B)\}, \sup_{b\in B}\{\mathrm{d}(b, A)\}\right\}.$$

By convention $\mathrm{H}_{\mathrm{d}}(\emptyset, \emptyset) = 0$ and, for $\emptyset \neq A \subset X$, $\mathrm{H}_{\mathrm{d}}(A, \emptyset) = \infty$. The function H_{d} is called the Hausdorff distance induced by d. In our case, since we will be working in $(\mathbb{R}^2, \mathrm{d})$, d being the usual Euclidean distance, we simplify the notation writing $\mathrm{H}(A, B)$.

1.3. Approximate parametrization problem and its applications

Many applications use algebraic curves and surfaces in their development. Examples of this situation can be found in computer aided design, computer graphics, geometric modeling, computer numerical control or pattern recognition (see [28,29]). Also, curves and surfaces are of interest in solving differential equations (see [30,31]) and diophantine equations (see [32]), in modeling lens for cameras (see [33]), shape symmetry detection (see [34]), or in the automatic determination of geometric loci in dynamical geometry (see [35]). Moreover, depending on the problem one uses different representations of the curve or the surface, namely: implicit or parametric. Usually computer graphics uses both types of representations, while computer aided geometric design, although also uses implicit representations, tends to focus on the parametric one. Among all parametric representations, rational parametrizations (i.e. parametrizations given by rational functions) are most often used, since they can be easily included into standard CAD systems.

However, although implicit representations are always available, rational parametric representations are not always possible; one may extend the class of rational curves and surfaces (i.e. curves and surfaces admitting a rational parametrization) to the class of curves and surfaces parametrizable by fractions of nested radicals of polynomials. Moreover, there exist exact algorithms for computing such parametrizations (see e.g. [36] for the rational case, and [37,38] for the radical case). Nevertheless, this extension is still small; for instance, rational curves are those of genus 0 and radical curves are those of genus at most 6. So the use of alternative approaches, as approximate techniques, is unavoidable. Some approximate techniques, as ours, provide exact rational parametrizations, if they exist, others generate piecewise rational parametrizations, in both cases, under a given precision.

In any case, at least to our knowledge, the existing algorithms focus on the computation of one particular parametrization, satisfying certain criteria, but they do not describe the set of all possible answers. Knowing this set will increase the applicability of the approach, since one may try to find the best answer among all satisfying the required criteria. This is the aim of this paper.

To finish this subsection, let us try to illustrate our claims by an example of a potential application of our contribution; the aim of this example is not to show a theoretical solution to a problem, but to illustrate how our contribution might help

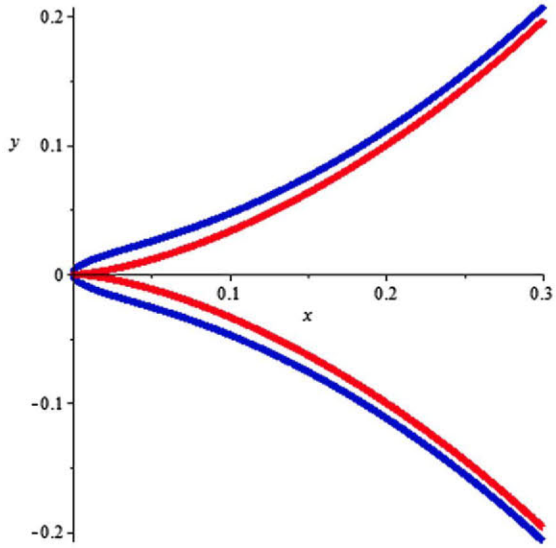

Fig. 2. Curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ in the example on offsets.

with how to approach it. Offset curves and surfaces are a very important geometric construction, used in many applications in computer aided geometry design (see [28,29]); specially rational offsets. In [39] it is shown that a necessary and sufficient condition for an offset to be rational is that the original curve is rational, and that the normal vector associated to a parametrization of the curve has a rational norm. Now, let us assume that we are given a non-rational curve $\mathcal{C}$, and we look for a rational curve $\overline{\mathcal{C}}$ close to $\mathcal{C}$ whose offsets are rational. If we apply directly an approximate parametrization algorithm we will get a rational curve close to $\mathcal{C}$ but we will not control the rationality of its offsets. Therefore, if we could describe all solutions, or infinitely many, we could check the existence of a suitable rational curve $\overline{\mathcal{C}}$ with rational offsets. Let $\mathcal{C}$ be defined by $x^3 + xy^2 - y^2 + \frac{1}{100}x = 0$. This is a Hausdorff divisor (see Definition 4.1) and hence our method is applicable. Furthermore, our method, using the origin as a double point, outputs the following family of infinitely many rational curves, all of them being at finite Hausdorff distance to $\mathcal{C}$,

$$x^3\lambda_4 + xy^2\lambda_4 + x^2\lambda_3 + xy\lambda_2 + y^2\lambda_1.$$

For most specializations of the parameters λ_i one would get a rational curve without rational offsets, however, for $\{\lambda_4 = 1, \lambda_1 = -1, \lambda_2 = 0, \lambda_3 = 0\}$ one gets the curve $\overline{\mathcal{C}}$, defined by $x^3 + y^2x - y^2 = 0$, that is rational and whose offsets are rational; in Fig. 2 one can see the input $\mathcal{C}$ and the output curve $\overline{\mathcal{C}}$.

1.4. Main contributions of the paper

In general terms, the main contribution of the paper is the development of a theory from where, and under the assumption that the given curve has as many different points at infinity as degree, infinitely many possible solutions of the approximate parametrization problem are described. From this analysis, one may determine an optimal or almost optimal (under certain given additional criteria) solution of the problem. For instance, one may try to provide a rational parametrization with small height (i.e. integer coefficients with small absolute value), or with a Hausdorff distance smaller than a given tolerance, or satisfying certain additional geometric features as having particular ramification points, type of singularities, tangencies, etc. More precisely, the main contributions of the paper can be summarized as follows:

- Theorem 6.4, in [25], gives conditions for the Hausdorff distance between two real algebraic curves to be finite. Based on this, we introduce the notion of Hausdorff divisor, we study the dimension and irreducibility of its linear associated system of curves (it is a projective linear subspace), and we prove that all irreducible real curves belonging to the linear system are at finite Hausdorff distance among them (see Theorem 2.11).
- In addition, we introduce the notion of rational Hausdorff divisor, we also study the dimension and irreducibility, and we prove that all irreducible real curves in the linear associated system are parametrizable by means of rational functions and are at finite Hausdorff distance (see Theorem 3.5). Therefore, we describe a projective linear subspace where all (irreducible) elements are solutions to the approximate parametrization problem.
- In a second stage, we identify the linear system of a rational Hausdorff divisor with a plane curve over the algebraic closure of a (in general) transcendental extension of $\mathbb{C}$. This curve is shown to be rational and we provide algorithms to parametrize it over simpler subfields (see Theorem 3.14 and its corollaries). This implies that we provide a generic answer to the approximate parametrization problem; that is, a rational parametrization curve with coefficients depending polynomially on a finite set of parameters.

- We introduce the notion of Hausdorff curve, that essentially requires that the curve has as many different points at infinity as degree.
- Furthermore, we prove that every irreducible Hausdorff curve can always be parametrized with a generic parametrization having coefficients depending on as many parameters as degree; so, with as many degrees of freedom as the degree of the curve (see Theorem 4.4 and its corollaries).

Therefore, the previous results provide an alternative method to the algorithm in [22] that is applicable to a wider family of curves and that provides, not one, but infinitely many solutions to the problem. We do not present any systematic study of how to proceed to choose an optimal (under a given criterion) solution from the set of infinitely many provided solutions; this is left as future research. Here instead, with an example, we illustrate the potential applicability of the method.

1.5. Structure of the paper

The paper is structured in 4 main sections. The first one (Section 2) is devoted to develop the new notion of Hausdorff divisor and to establish its main properties. For this purpose, a preliminary subsection on divisors is included. In Section 3, we focus on the combination of rational divisors with Hausdorff divisors as well as on the generic parametrization of the associated curve to the divisor, analyzing the field of parametrization; these parametrizations are, in fact, approximate parametrization for every real irreducible curve in the associated system to the Hausdorff divisor. Section 4 deals with the application of the previously developed results to the approximate parametrization of Hausdorff divisors. The paper ends with a section (Section 5) of conclusions where we, in addition, comment on some future directions of research.

The computations and pictures, in this paper, have been performed with the mathematical software Maple 17 (see maplesoft documentation center).

2. Algebraic curves and divisors

In this section we first recall the notion of divisor and its relation to algebraic curves. Then, we introduce the new notion of Hausdorff divisor and we study its main properties. In the last part, we analyze 2-degree Hausdorff divisors.

2.1. Preliminaries on divisors and linear systems of curves

Throughout this paper we denote by $\mathbb{P}^2(\mathbb{C})$ the projective plane over the field $\mathbb{C}$ of complex numbers. Let us start recalling the notion of divisor. A divisor is, intuitively speaking, a way of describing finite collections of points in $\mathbb{P}^2(\mathbb{C})$ with assigned (maybe negative) multiplicities. More precisely, a divisor in $\mathbb{P}^2(\mathbb{C})$ is a formal expression

$$\sum_{i=1}^{m} s_i P_i$$

where $s_i \in \mathbb{Z}$ and P_i are different points in $\mathbb{P}^2(\mathbb{C})$; if s_i are all non-negative integers, the divisor is called effective. In this paper, we are only interested in effective divisors; non-effective divisors are used when poles of rational functions need to be analyzed. We define the degree of the divisor $D = \sum_{i=1}^{m} s_i P_i$ as the number

$$\deg(D) := \sum_{i=1}^{m} s_i.$$

Let n be a positive integer, and let $\mathcal{C}$ be a projective algebraic plane curve of degree n. $\mathcal{C}$ will be defined by a homogeneous polynomial $F(x, y, z) \in \mathbb{C}[x, y, z]$. We can identify $\mathcal{C}$ with the projective point given by its coefficients after fixing a term order. For instance, if $n = 2$, let us fix e.g. the order $y^2 < xy < x^2 < yz < xz < z^2$, then the circle $x^2 + y^2 - z^2 = 0$ is seen as $(1 : 0 : 1 : 0 : 0 : -1) \in \mathbb{P}^5(\mathbb{C})$ and any other conic is a point in $\mathbb{P}^5(\mathbb{C})$; e.g. $(1 : 0 : -1 : 0 : 0 : -1)$ is the hyperbola $y^2 - x^2 - z^2$.

In this situation, the set of all projective curves of degree n is identified with the projective space $\mathbb{P}^{\ell-1}(\mathbb{C})$, where $\ell = (n + 1)(n + 2)/2$. Now, a linear system of curves of degree n is a linear subspace of $\mathbb{P}^{\ell}(\mathbb{C})$. Considering parametric equations of the linear system one can see the system as a homogeneous form whose coefficients are those equations. This form is called the defining polynomial of the linear system. For instance, let $n = 1$ and $x < y < z$, then the linear subspace, of degree 1, parametrized as $(\lambda : \mu : 0)$ corresponds to the form $\lambda x + \mu y$ that defines the pencil of projective lines passing through the origin $(0 : 0 : 1)$. We, indistinctly, will see the linear system as a projective linear subspace or as a homogeneous polynomial.

Associated with an effective divisor, one can consider a linear system of curves for a positive integer n, big enough. More precisely, let $D = \sum_{i=1}^{m} s_i P_i$ be an effective divisor, then we consider the set of all projective curves of degree n passing through P_i with multiplicity, at least, s_i, for $i = 1, \ldots, m$. Observe that these requirements are linear conditions on the coefficients of the generic form of degree n; see Example 2.3 for details on how to compute this. Therefore, this set is a linear system of curves. We denote it by $\mathcal{H}(n, D)$.

A natural question is the analysis of the dimension of a linear system. In general, if D is an effective divisor it holds (see Theorem 2.59 in [36]) that

$$\dim(\mathcal{H}(n, D)) \geq \frac{n(n+3)}{2} - \sum_{i=1}^{m} \frac{s_i(s_i+1)}{2}. \tag{1}$$

One may also consider the notion of divisor in general position (see Section 2.4 in [36]).

Let D be an effective divisor such that $\mathcal{H}(n, D) \neq \emptyset$, and let $H(\Lambda, x, y, z)$ be its defining polynomial, where Λ is a tuple of parameters. For each specialization Λ_0 of Λ, taking values in $\mathbb{C}$, we get a projective curve, namely the curve defined by $H(\Lambda_0, x, y, z)$. Alternatively, we can see $\mathcal{H}(n, D)$ as a projective plane curve over the algebraic closure of $\mathbb{C}(\Lambda)$ defined by $H(\Lambda, x, y, z)$. This motivates the following definition.

Definition 2.1. Let D be an effective divisor, let $\emptyset \neq \overline{\mathcal{H}} \subseteq \mathcal{H}(n, D)$, and let $\overline{H}(\overline{\Lambda}, x, y, z)$ be the defining polynomial of $\overline{\mathcal{H}}$. The projective plane curve defined by $\overline{H}(\overline{\Lambda}, x, y, z)$, over the algebraic closure of $\mathbb{C}(\overline{\Lambda})$, is called the projective algebraic curve associated to $\overline{\mathcal{H}}$ and we denote it by $\mathrm{Curve}(\overline{\mathcal{H}})$; in general we will identify $\overline{\mathcal{H}}$ and $\mathrm{Curve}(\overline{\mathcal{H}})$. •

Example 2.2. Let $D = (0 : 0 : 1)$. We consider $\mathcal{H}(1, D)$. The defining polynomial of $\mathcal{H}(1, D)$ is (here $\Lambda = (\lambda_1, \lambda_2)$)

$$H(\Lambda, x, y, z) = \lambda_1 x + \lambda_2 y.$$

Therefore, $\mathcal{H}(1, D)$ consists in all lines in $\mathbb{P}^2(\mathbb{C})$ passing through $(0 : 0 : 1)$. However, $\mathrm{Curve}(\mathcal{H}(1, D))$ is a particular line in $\mathbb{P}^2(\overline{\mathbb{C}(\Lambda)})$ namely the line $y = -\lambda_1/\lambda_2 x$, where $\overline{\mathbb{C}(\Lambda)}$ is the algebraic closure of $\mathbb{C}(\Lambda)$. Therefore, the pencil of lines $H(\Lambda, x, y, z)$ is identified with the line $y = -\lambda_1/\lambda_2 x$. □

Example 2.3. Let $D = (1 : \pm\mathrm{i} : 0) + (0 : \pm 1 : 1)$, where i is the imaginary unit. We consider $\mathcal{H}(2, D)$. In order to compute the linear system we consider a generic form of degree 2, namely, $H(\Lambda, x, y, z) = x^2\lambda_6 + xy\lambda_5 + xz\lambda_4 + y^2\lambda_3 + yz\lambda_2 + z^2\lambda_1$. Now, we get the linear conditions

$$\begin{cases} H(\Lambda, 1, \mathrm{i}, 0) = 0 \\ H(\Lambda, 1, -\mathrm{i}, 0) = 0 \\ H(\Lambda, 0, 1, 1) = 0 \\ H(\Lambda, 0, -1, 1) = 0 \end{cases} \Rightarrow \begin{cases} \lambda_6 + i\lambda_5 - \lambda_3 = 0 \\ \lambda_6 - i\lambda_5 - \lambda_3 = 0 \\ \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ \lambda_1 - \lambda_2 + \lambda_3 = 0. \end{cases}$$

Solving the system of equations we get that the defining polynomial of $\mathcal{H}(2, D)$ is

$$H(\Lambda, x, y, z) = x^2\lambda_6 + xz\lambda_4 + y^2\lambda_6 - z^2\lambda_6,$$

that describes a pencil of conics. Indeed since D contains the cyclic points, almost all conics in the pencil $\mathcal{H}(2, D)$ are circles; note that for $\lambda_6 = 0$, $\lambda_4 \neq 0$ one gets $xz = 1$. On the other hand, H can be expressed as

$$H(\Lambda, x, y, z) = \lambda_6 \left(\left(x + \frac{\lambda_4 z}{2\lambda_6} \right)^2 + y^2 - \frac{z^2 \left(4\,\lambda_6{}^2 + \lambda_4{}^2 \right)}{4\lambda_6{}^2} \right).$$

Thus, $\mathrm{Curve}(\mathcal{H}(2, D))$ is the circle of affine center $(-\frac{\lambda_4}{2\lambda_6}, 0)$ and radius $\frac{\sqrt{4\lambda_6^2+\lambda_4^2}}{2\lambda_6}$. □

2.2. Hausdorff divisors

In this subsection, we introduce the new concept of Hausdorff divisor. These divisors are related to the structure at infinity of algebraic curves, and we will see that they are useful to solve the approximate parametrization problem.

Definition 2.4. We say that a divisor $\sum_{i=1}^{m} s_i P_i$ is a Hausdorff divisor if, for all $i \in \{1, \ldots, m\}$, $s_i = 1$ and P_i is of the form $(a : b : 0) \in \mathbb{P}^2(\mathbb{C})$.

Let D be an n-degree Hausdorff divisor, the linear system $\mathcal{H}(n, D)$ is called the Hausdorff linear system associated to D. •

Using inequality (1), see Section 2.1, the following result on the dimension of Hausdorff linear systems is deduced.

Proposition 2.5. *Let D be an n-degree Hausdorff divisor. Then*

$$\dim(\mathcal{H}(n, D)) \geq \frac{n(n+1)}{2}.$$

Our main goal will be to parametrize the projective algebraic curve associated to a linear system. This implies that the curve has to be irreducible, and hence the next notion appears naturally.

Definition 2.6. Let D be an effective divisor such that $\mathcal{H}(n, D) \neq \emptyset$. We say that D is irreducible if $\mathsf{Curve}(\mathcal{H}(n, D))$ is irreducible; that is, if the defining polynomial $H(\Lambda, x, y, z)$ of $\mathcal{H}(n, D)$ is irreducible over the algebraic closure of $\mathbb{C}(\Lambda)$. •

Example 2.7. Let $D = 2(1 : 0 : 0)$. The defining polynomial of $\mathcal{H}(2, D)$ is

$$H(\Lambda, x, y, z) = \lambda_1 z^2 + \lambda_2 yz + \lambda_3 y^2, \quad \text{where } \Lambda = (\lambda_1, \lambda_2, \lambda_3).$$

This polynomial is irreducible over $\mathbb{C}$. However, over the algebraic closure $\overline{\mathbb{C}(\Lambda)}$ of $\mathbb{C}(\Lambda)$, H factors as

$$\begin{aligned} H(\Lambda, x, y, z) &= \lambda_1 \left(z + \frac{1}{2}\frac{\lambda_2 y}{\lambda_1}\right)^2 + \left(\lambda_3 - \frac{1}{4}\frac{\lambda_2{}^2}{\lambda_1}\right) y^2 \\ &= \left(\sqrt{\lambda_1}\left(z + \frac{1}{2}\frac{\lambda_2 y}{\lambda_1}\right) + \frac{\mathrm{i}}{2}\sqrt{4\lambda_3 - \frac{\lambda_2{}^2}{\lambda_1}}\, y\right) \cdot \left(\sqrt{\lambda_1}\left(z + \frac{1}{2}\frac{\lambda_2 y}{\lambda_1}\right) - \frac{\mathrm{i}}{2}\sqrt{4\lambda_3 - \frac{\lambda_2{}^2}{\lambda_1}}\, y\right). \end{aligned}$$

Thus D is reducible or, equivalently $\mathsf{Curve}(\mathcal{H}(2, D))$ is reducible; indeed, it is a pair of lines. Note that D is not Hausdorff. □

In Example 2.7 we have been able to factor H over the algebraic closure $\overline{\mathbb{C}(\Lambda)}$ of $\mathbb{C}(\Lambda)$. This was easy because we dealt with a 2-degree polynomial. Nevertheless, in general, to our knowledge, there exists no algorithm to factor polynomials over $\overline{\mathbb{C}(\Lambda)}$. However in [40, Theorems 5.5.2 and 5.5.3], sufficient conditions are given to reduce the irreducibility over $\overline{\mathbb{C}(\Lambda)}$ to the irreducibility over $\mathbb{C}$; note that for polynomials, with coefficients in a computable subfield of $\mathbb{C}$, there exist absolute factorization algorithms. The fundamental fact is that Hausdorff divisors (see Theorem 2.9) are always irreducible, and that for subsystems of associated systems to Hausdorff divisors the irreducibility over $\overline{\mathbb{C}(\Lambda)}$ is equivalent to the irreducibility over $\mathbb{C}$ (see Corollary 2.10), and hence there is no computational obstacle for our purposes.

In the next theorem, we study the irreducibility of $\mathsf{Curve}(\mathcal{H}(n, D))$, when D is a Hausdorff divisor; observe that a Hausdorff linear system is never empty (see Proposition 2.5), and hence $\mathsf{Curve}(\mathcal{H}(n, D))$ always exists. We start with the following lemma.

Lemma 2.8. *Let D be an n-degree Hausdorff divisor. The defining polynomial of $\mathcal{H}(n, D)$ is irreducible over $\mathbb{C}$.*

Proof. Let $H(\Lambda, x, y, z)$ be the defining polynomial of $\mathcal{H}(n, D)$ and let $\mathbb{F}$ be the algebraic closure of $\mathbb{C}(\Lambda)$. If H factors over $\mathbb{C}$, with factors depending not only on Λ, then all curves in the linear system are reducible. So, to prove the statement, we find a specific irreducible projective curve in $\mathcal{H}(n, D)$. Let us assume that $D = \sum_{i=1}^{n} (a_i : b_i : 0)$. Let $(a : b : 0)$ be different to all points in D. We consider the projective curve $\mathcal{C}$ defined by

$$F(x, y, z) = z(bx - ay)^{n-1} - \prod_{i=1}^{n} (b_i x - a_i y).$$

Since $(a : b : 0)$ is different to $(a_i : b_i : 0)$, F is irreducible and clearly $\mathcal{C} \in \mathcal{H}(n, D)$. □

Theorem 2.9. *Let D be an n-degree Hausdorff divisor. Then, D is irreducible.*

Proof. Let $H(\Lambda, x, y, z)$, $\mathcal{H}(n, D)$ and $\mathbb{F}$ be as in the proof of Lemma 2.8. We may assume w.l.o.g. that H is monic w.r.t. y (this is equivalent to $(0 : 1 : 0) \notin D$): indeed, if it is not the case, we can always perform a projective change of coordinates over $\mathbb{C}$, such that $(0 : 1 : 0) \notin D$ and D stays Hausdorff; then the irreducibility of H over $\mathbb{F}$ is preserved. Since D is Hausdorff, by Lemma 2.8, x does not divide H. Therefore, H is irreducible over $\mathbb{F}$ iff $h(y, z) = H(1, y, z)$ is irreducible over $\mathbb{F}$. $h(y, z)$ is monic in y. Moreover, since D is Hausdorff, $h(y, 0)$ is square-free. Therefore, using Theorem 5.5.2 in [40] (note that since D is Hausdorff, the algebraic element β, in the statement of Theorem 5.5.2, can be taken as a complex number), we have that h is irreducible over $\mathbb{F}$ iff h is irreducible over $\mathbb{C}$. Now, the result follows from Lemma 2.8. □

Corollary 2.10. *Let D be an n-degree Hausdorff divisor, and let $\overline{H}(\overline{\Lambda}, x, y, z)$ be the defining polynomial of a non-empty linear subsystem $\overline{\mathcal{H}} \subseteq \mathcal{H}(n, D)$. Then, $\overline{H}$ is irreducible over the algebraic closure of $\mathbb{C}(\overline{\Lambda})$ if and only if $\overline{H}$ is irreducible over $\mathbb{C}$; that is, $\mathsf{Curve}(\overline{\mathcal{H}})$ is irreducible if and only if $\overline{H}$ is irreducible over $\mathbb{C}$.*

Proof. Let left–right implication is trivial. The other implication follows as the proof of Theorem 2.9 but using the irreducibility of $\overline{H}$ over $\mathbb{C}$, instead of Lemma 2.8. □

The next theorem states the main result on Hausdorff divisors. For this purpose, if $\mathcal{C}$ is the projective algebraic curve defined by the form $F(x, y, z)$, and it is different to the line at infinity $z = 0$, we denote by $\mathcal{C}_a$ the affine algebraic curve defined by $F(x, y, 1)$. Furthermore, for an affine algebraic curve $\mathcal{C}_a$ we denote by $\mathcal{C}_a^{\infty}$ the points at infinity of $\mathcal{C}_a$. We recall that an affine curve is real if it contains infinitely many real points.

Theorem 2.11. *Let D be an n-degree Hausdorff divisor. For every two real irreducible curves $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{H}(n, D)$, such that $\deg(\mathcal{C}_{i,a}) = n$, it holds that*

$$\mathrm{H}(\mathcal{C}_{1,a} \cap \mathbb{R}^2, \mathcal{C}_{2,a} \cap \mathbb{R}^2) < \infty.$$

Proof. Let $D = \sum_{i=1}^{n} P_i$. Since $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{H}(n, D)$ then $\mathcal{C}_{1,a}^{\infty} = \mathcal{C}_{2,a}^{\infty} = \{P_1, \ldots, P_n\}$. Moreover, $\mathrm{card}(\mathcal{C}_{1,a}^{\infty}) = \mathrm{card}(\mathcal{C}_{2,a}^{\infty}) = \deg(\mathcal{C}_{1,a}) = \deg(\mathcal{C}_{2,a})$. Now, the result follows from Theorem 6.4 in [25].

In the following we find conditions on the Hausdorff divisor $D = \sum_{i=1}^{n} P_i$ such that $\mathcal{H}(n, D)$ contains curves verifying the hypotheses of Theorem 2.11. For this purpose, we will use the concept of family of conjugate points that can be introduced as follows; see Definition 3.15 in [36] for further details. Let $\mathbb{K}$ be a subfield of $\mathbb{C}$, e.g. $\mathbb{K} = \mathbb{R}$, then a finite family of points is $\mathbb{P}^2(\mathbb{C})$ is $\mathbb{K}$ -conjugate if it can be expressed as

$$\{(p_1(t) : p_2(t) : p_3(t)) \mid m(t) = 0\}$$

where $p_i, m \in \mathbb{K}[t]$ and $\gcd(p_1, p_2, p_3) = 1$; for instance, the points in $\mathcal{F} := \{(\pm \mathrm{i} : 1 : 0)\}$ are $\mathbb{Q}$-conjugated since $\mathcal{F} = \{(t : 1 : 0) \mid t^2 + 1 = 0\}$.

Let $\mathcal{C} \in \mathcal{H}(n, D)$ be such that $\deg(\mathcal{C}_a) = n$, and $\mathcal{C}_a$ is real and irreducible. Let $F(x, y, z)$ be the defining polynomial of $\mathcal{C}$. Then, $\{P_1, \ldots, P_n\}$ is the family of conjugate points $\{(t : h : 0) | F(t, h, 0) = 0\}$. Moreover, since $\mathcal{C}$ is real, then F is a real polynomial (see Lemma 7.2 in [36]), and thus the family is $\mathbb{R}$-conjugated. This motivates the following definition.

Definition 2.12. Let $\mathbb{K}$ be a subfield of $\mathbb{C}$. We say that a Hausdorff divisor $D = \sum_{i=1}^{m} P_i$ is $\mathbb{K}$ -definable if $\{P_1, \ldots, P_m\}$ is a $\mathbb{K}$-conjugate family of points. •

In the next examples, we illustrate the notion of $\mathbb{R}$-definability and Theorem 2.11.

Example 2.13. We consider the Hausdorff divisor (i is the imaginary unit)

$$\begin{aligned} D = & \left(\frac{1}{2}\sqrt{2} + \frac{1}{2}\mathrm{i}\sqrt{2} + 1 : \mathrm{i} : 0\right) + \left(-\frac{1}{2}\sqrt{2} + \frac{1}{2}\mathrm{i}\sqrt{2} + 1 : -\mathrm{i} : 0\right) \\ & + \left(-\frac{1}{2}\sqrt{2} - \frac{1}{2}\mathrm{i}\sqrt{2} + 1 : \mathrm{i} : 0\right) + \left(\frac{1}{2}\sqrt{2} - \frac{1}{2}\mathrm{i}\sqrt{2} + 1 : -\mathrm{i} : 0\right). \end{aligned}$$

D can be expressed as $D = \sum (\alpha + 1 : \alpha^2 : 0)$, where $\alpha^4 + 1 = 0$. Therefore, D is $\mathbb{R}$-definable. □

Example 2.14. We consider the 4-degree Hausdorff divisor

$$D = (1 : 1 : 0) + (-1 : 1 : 0) + (0 : 1 : 0) + (1 : 0 : 0).$$

The defining polynomial of $\mathcal{H}(4, D)$ is

$$\begin{aligned} H = & \lambda_{11}z^4 + \lambda_{10}yz^3 + \lambda_9 y^2z^2 + \lambda_8 y^3 z + \lambda_7 xz^3 + \lambda_6 xyz^2 + \lambda_5 xy^2 z - \lambda_1 xy^3 \\ & + \lambda_4 x^2z^2 + \lambda_3 x^2yz + \lambda_2 x^3 z + \lambda_1 x^3 y. \end{aligned}$$

Note that the number of parameters λ_i is 11, and hence $\dim(\mathcal{H}(4, D)) = 10$; compare to Proposition 2.5. In Fig. 3 one may see 3 different curves in $\mathcal{H}(4, D)$. Observe that all of them have asymptotes in the direction of the vectors $(1, 1)$, $(-1, 1)$, $(1, 0)$, $(0, 1)$. □

2.3. Conics: 2-degree real definable Hausdorff divisors

In this subsection we analyze the 2-degree $\mathbb{R}$-definable Hausdorff divisors. We distinguish two cases: first the two points of the divisor are real, and second the two points are complex in which case they have to be conjugated because of the $\mathbb{R}$-definability.

[Real points: the non-compact case]. We consider 2-degree Hausdorff divisors with real points. We distinguish several cases. We start with $D = (1 : 0 : 0) + (0 : 1 : 0)$. The defining polynomial of $\mathcal{H}(2, D)$ is $H = a_{0,0}z^2 + a_{0,1}yz + a_{1,0}xz + a_{1,1}xy$. We may assume w.l.o.g. that $a_{1,1} \neq 0$, since otherwise for all $\mathcal{C}$, in the linear system, $\deg(\mathcal{C}_a) = 1$. Then, $H(x, y, 1)$ can be expressed as

$$H(x, y, 1) = \left(x + \frac{a_{0,1}}{a_{1,1}}\right)\left(y + \frac{a_{1,0}}{a_{1,1}}\right) + \frac{a_{0,0}}{a_{1,1}} - \frac{a_{0,1}a_{1,0}}{{a_{1,1}}^2}.$$

Now, observe that all real irreducible affine curves derived from the system are hyperbolas with parallel asymptotes, indeed with direction vectors $(1, 0)$ and $(0, 1)$ (compare to Theorem 3 in [41] or Lemma 6.1 in [22]), and hence Theorem 2.11 holds.

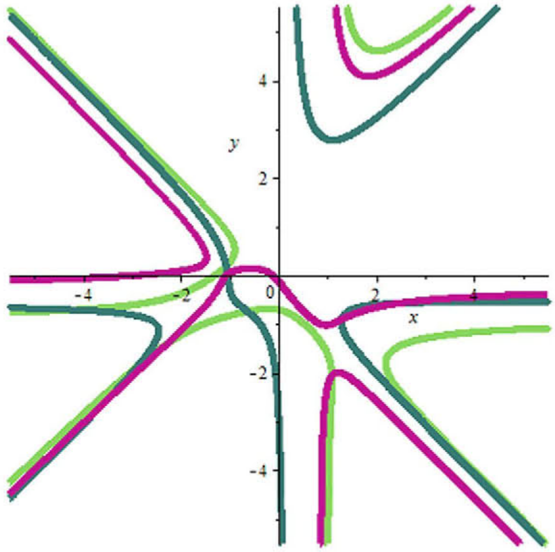

Fig. 3. Some curves in $\mathcal{H}(4, D)$ in Example 2.14.

Second, we take $D = (a : 1 : 0) + (b : 1 : 0)$, with $a, b \in \mathbb{R}, a \neq b$. The defining polynomial of $\mathcal{H}(2, D)$ is $H = a_{0,0}z^2 + a_{0,1}yz + ba_{2,0}ay^2 + a_{1,0}xz - a_{2,0}axy - xya_{2,0}b + a_{2,0}x^2$. We may assume w.l.o.g. that $a_{2,0} \neq 0$, since otherwise for all $\mathcal{C}$, in the linear system, $\deg(\mathcal{C}_a) = 1$. Then, $H(x, y, 1)$ can be expressed as

$$\frac{1}{4}\left(2x + \frac{a_{1,0}}{a_{2,0}} - ay - by\right)^2 - \frac{1}{4}\left((a-b)\,y - \frac{\Delta}{(a-b)\,a_{2,0}{}^2}\right)^2 + \frac{1}{4}\frac{4a_{0,0}a_{2,0} - a_{1,0}{}^2}{a_{2,0}{}^2} + \frac{1}{8}\frac{\Delta^2}{(a-b)^2\,a_{2,0}{}^4},$$

where $\Delta = a_{2,0}\left(2a_{0,1} + a_{1,0}a + a_{1,0}b\right)$. Now, observe that all real irreducible affine curves derived from the system are hyperbolas with parallel asymptotes, indeed with direction vectors $(a, 1)$ and $(b, 1)$ (compare to Theorem 3 in [41] or Lemma 6.1 in [22]), and hence Theorem 2.11 holds.

Third, we take $D = (1 : 0 : 0) + (b : 1 : 0)$, with $b \in \mathbb{R}, b \neq 0$. The defining polynomial of $\mathcal{H}(2, D)$ is $H = a_{0,0}z^2 + a_{0,1}yz - a_{1,1}by^2 + a_{1,0}xz + a_{1,1}xy$. We may assume w.l.o.g. that $a_{1,1} \neq 0$, since otherwise for all $\mathcal{C}$, in the linear system, $\deg(\mathcal{C}_a) = 1$. Then, $H(x, y, 1)$ can be expressed as

$$\left(y - \frac{a_{0,1}}{2a_{1,1}b} - \frac{x}{2b}\right)^2 - \frac{1}{4b^2}\left(x + 2\frac{a_{1,0}b}{a_{1,1}} + \frac{a_{0,1}}{a_{1,1}}\right)^2 - \frac{1}{b^2}\left(\frac{ba_{0,0}}{a_{1,1}} + \frac{a_{0,1}{}^2}{a_{1,1}{}^2}\right) + \frac{1}{4b^2}\left(2\frac{a_{1,0}b}{a_{1,1}} + \frac{a_{0,1}}{a_{1,1}}\right)^2.$$

Now, observe that all real irreducible affine curves derived from the system are hyperbolas with parallel asymptotes, indeed with direction vectors $(1, 0)$ and $(b, 1)$ (compare to Theorem 3 in [41] or Lemma 6.1 in [22]), and hence Theorem 2.11 holds.

[Complex points: the compact case] Since both points have to be complex and conjugated, we can assume w.l.o.g. that D is of the form $D = (a + \mathrm{i} : 1 : 0) + (a - \mathrm{i} : 1 : 0)$, where i is the imaginary unit. The defining polynomial of $\mathcal{H}(2, D)$ is $H = a_{0,0}z^2 + a_{0,1}yz + y^2a_{2,0}a^2 + y^2a_{2,0} + a_{1,0}xz - 2a_{2,0}axy + a_{2,0}x^2$. We may assume w.l.o.g. that $a_{2,0} \neq 0$, since otherwise for all $\mathcal{C}$, in the linear system, $\deg(\mathcal{C}_a) = 1$. Then, $H(x, y, 1)$ can be expressed as

$$\left(x + \frac{a_{1,0}}{2a_{2,0}} - ay\right)^2 + \left(y + \frac{a_{0,1} + a_{1,0}a}{2a_{2,0}}\right)^2 - \frac{\Sigma}{4a_{2,0}{}^2}$$

where $\Sigma = -4a_{0,0}a_{2,0} + a_{1,0}{}^2 + a_{0,1}{}^2 + 2a_{0,1}a_{1,0}a + a_{1,0}{}^2a^2$. So, if $a \neq 0$ we get ellipses and for $a = 0$ we get circles (note that for $a = 0$ the divisor is defined by the cyclic points)

$$\left(x + \frac{a_{1,0}}{2a_{2,0}}\right)^2 + \left(y + \frac{a_{0,1}}{2a_{2,0}}\right)^2 - \frac{-4a_{0,0}a_{2,0} + a_{1,0}{}^2 + a_{0,1}{}^2}{4a_{2,0}{}^2}.$$

In both cases, the statement in Theorem 2.11 clearly holds.

3. Approximate parametrization of algebraic curves and divisors

In Theorem 2.11 we have seen that in the linear system $\mathcal{H}(n, D)$, of a Hausdorff divisor D, all irreducible real curves are at finite Hausdorff distance. Now, let us assume that we are given a curve $\mathcal{C} \in \mathcal{H}(n, D)$ and we want to parametrize it approximately. For this purpose, we find a subsystem $\overline{\mathcal{H}}$ of $\mathcal{H}(n, D)$, such that all irreducible real curves in $\overline{\mathcal{H}}$ are rational and at finite distance of $\mathcal{C}$. This yields to the notion of rational Hausdorff divisor. Moreover, parametrizing $\mathsf{Curve}(\overline{\mathcal{H}})$ one gets the family of all approximate parametrizations of $\mathcal{C}$ in $\overline{\mathcal{H}}$. The section is structured in two subsections. The first focuses on the notion of rational Hausdorff divisor, and the second on the parametrization of linear systems associated to rational Hausdorff divisors.

We start this section recalling briefly the concept of rational curve. An algebraic curve is called rational if it can be parametrized by means of rational functions; in other words, if $F(x, y, z)$ is the homogeneous polynomial defining a projective curve $\mathcal{C}$, then $\mathcal{C}$ is rational if there exist three polynomials $p_1(t), p_2(t), p_3(t)$, not all constant, such that $\gcd(p_1, p_2, p_3) = 1$, and $F(p_1(t), p_2(t), p_3(t)) = 0$. In this case, $(p_1(t), p_2(t), p_3(t))$ is called a rational parametrization of $\mathcal{C}$. If $\mathcal{C}$ is not the line at infinity $z = 0$, we usually write the parametrization as $(p_1(t)/p_3(t), p_2(t)/p_3(t), 1)$. The rationality of a curve can be deduced from its genus. An irreducible curve is rational if and only if its genus is 0.

The genus, intuitively speaking, measures the difference between the maximum of singularities the curve may have and the actual number of them. More precisely, the genus is given by the formula

$$\text{genus}(\mathcal{C}) = \frac{(\deg(\mathcal{C}) - 1)(\deg(\mathcal{C}) - 2)}{2} - \sum_{P \in \mathcal{C}} \frac{\text{mult}(\mathcal{C}, P)(\text{mult}(\mathcal{C}, P) - 1)}{2}$$

where $\deg(\mathcal{C})$ denotes the degree of $\mathcal{C}$ (i.e. the degree of the form F), $\text{mult}(\mathcal{C}, P)$ denotes the multiplicity of $\mathcal{C}$ at P, and where the sum is taken also over the infinitely near, or neighboring, points (see Chapter 3 in [36] for further details). Note that if $\mathcal{C}$ is irreducible and has a point of multiplicity $(\deg(\mathcal{C}) - 1)$ then the genus is 0, and hence $\mathcal{C}$ is rational. Curves satisfying this particular case are called monomial curves.

3.1. Rational Hausdorff divisors

In this subsection, we introduce the notion of rational divisor or genus 0 divisor, and we combine it with the new concept of Hausdorff divisor. Similarly, one can consider the concept of genus g divisor but, here, we are only interested in the genus 0 case. The definition we give here focuses on singularities of ordinary type; i.e. all tangents at the point are different. The case of non-ordinary singularities can also be introduced; at the end of this section, we briefly comment how to do it and we illustrate these ideas in Example 3.13.

Definition 3.1. Let $n \in \mathbb{N}, n > 0$, and $D = \sum_{i=1}^{m} s_i P_i$ be an effective divisor. If $n \in \{1, 2\}$, we say that D is an n-rational divisor if $\deg(D) = 1$. If $n > 2$, we say that D is an n-rational divisor if $s_i > 1$ for $i = 1, \ldots, m$, and

$$(n - 1)(n - 2) = \sum_{i=1}^{m} s_i(s_i - 1).$$

If D is n-rational, and only contains a point, we say that D is an n-monomial divisor. •

Note that $D = P$ is a 1-monomial and a 2-monomial divisor. In general, for $n > 2$, $D = (n-1)P$ is an n-monomial divisor. On the other hand, for $n = 3$ the only possible rational divisors are monomial, i.e. $D = 2P$, while for $n > 3$ the situation is open to more possibilities; for instance, for $n = 4$, one has $D = 3P$ or $D = 2P_1 + 2P_2 + 2P_3$.

Rational divisors and rational curves are closely related. More precisely, let $\mathcal{C}$ be a curve of degree n and genus 0, and let D be the divisor associated to the singular locus $\text{Sing}(\mathcal{C})$ of $\mathcal{C}$, that is (for simplicity we assume that $\mathcal{C}$ has only ordinary singularities, see [36] for the general case),

$$D = \sum_{P \in \text{Sing}(\mathcal{C})} (\text{mult}(P, \mathcal{C}) - 1)P.$$

Since $\text{genus}(\mathcal{C}) = 0$, then D is $(n - 1)$-rational.

The singular locus, and hence the rational divisor, of a real irreducible plane curve can be decomposed as the union of conjugate singularities (see Section 3.3 in [36]; more particularly Corollary 3.23). We introduce the next definition.

Definition 3.2. Let $\mathbb{K}$ be a subfield of $\mathbb{C}$. We say that a rational divisor D is $\mathbb{K}$-definable if D can be expressed as

$$D = \sum_{i=1}^{m_1} s_1 P_{1,i} + \cdots + \sum_{i=1}^{m_k} s_k P_{k,i}$$

where $\{P_{j,1}, \ldots, P_{j,m_j}\}$ is a family of $\mathbb{K}$-conjugated points, for $j = 1, \ldots, k$. •

Observe that a monomial divisor is $\mathbb{R}$-definable if and only if the point in the divisor is real. The next results deal with the dimension.

Theorem 3.3. *Let D be an n-rational divisor, then* $\dim(\mathcal{H}(n, D)) \geq 3n - 1 - \deg(D)$.

Proof. It follows from inequality (1) and Definition 3.1. □

Corollary 3.4. *Let D be an n-monomial divisor, then* $\dim(\mathcal{H}(n, D)) \geq 2n$.

The main property on this type of divisors is the following.

Theorem 3.5. *Let D be an n-rational divisor.*

1. *Every irreducible curve in $\mathcal{H}(n, D)$ is rational.*
2. *If D is irreducible (see Definition 2.6), then* $\mathsf{Curve}(\mathcal{H}(n, D))$ *is rational.*

Proof. Since D is rational, if the curve is irreducible, its genus is zero. So the curve is rational. □

Our next step is to combine both notions, Hausdorff and rational divisor.

Definition 3.6. We say that an effective divisor D is an n-rational Hausdorff divisor if D can be expressed as

$$D = D_H + D_S$$

where D_H is n-degree Hausdorff, and D_S is n-rational and no point in D_S is on the line $z = 0$ (i.e. all points in D_S are affine). If both D_H, D_S are $\mathbb{K}$-definable, we say that D is $\mathbb{K}$-definable, where $\mathbb{K}$ is a subfield of $\mathbb{C}$. Given a rational Hausdorff divisor D, we denote by D_H and D_S the Hausdorff and the singular part of D, respectively. In addition, we say that $\mathcal{H}(n, D)$ is the rational Hausdorff linear space associated to D. •

Note that, since all points in D_H have to be at infinity and all points in D_S have to be affine, the decomposition $D_H + D_S$ is unique.

Now, we analyze $\mathcal{H}(n, D)$ where D is an n-rational Hausdorff divisor. First, we observe that every irreducible curve in $\mathcal{H}(n, D)$ is smooth at the line $z = 0$ and rational. Let us study the dimension. By Proposition 2.5 and Theorem 3.3, we get the following result.

Theorem 3.7. *Let $D = D_H + D_S$ be an n-rational Hausdorff divisor then*

$$\dim(\mathcal{H}(n, D)) \geq 2n - 1 - \deg(D_S).$$

Corollary 3.8. *If D is an n-monomial Hausdorff divisor, then* $\dim(\mathcal{H}(n, D)) \geq n$.

We illustrate the previous results by some examples.

Example 3.9. We consider the divisor $D = (1 : 1 : 0) + (-1 : 1 : 0) + (0 : 1 : 0) + (1 : 0 : 0) + 2(3 : -2 : 1) + 2(1 : 1 : 1) + 2(2 : 3 : 1)$. $D = D_H + D_S$ where

$$\begin{aligned} D_H &= (1 : 1 : 0) + (-1 : 1 : 0) + (0 : 1 : 0) + (1 : 0 : 0), \\ D_S &= 2(3 : -2 : 1) + 2(1 : 1 : 1) + 2(2 : 3 : 1). \end{aligned}$$

Note that D_H is a 4-degree Hausdorff divisor (indeed, it is the one in Example 2.14) and D_S is a 4-rational divisor (note that $\sum s_i(s_i - 1) = 2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 = 3 \cdot 2 = (n-1)(n-2)$). So, D is a 4-rational Hausdorff divisor, in fact $\mathbb{R}$-definable. To compute the defining polynomial of $\mathcal{H}(4, D)$ we consider the generic expression of a 4-degree homogeneous polynomial, in the variables $\{x, y, z\}$, with undetermined coefficients $\Lambda = (\lambda_i)$; say, $H(\Lambda, x, y, z)$. Then, D imposes the conditions (here $\nabla H = (\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}, \frac{\partial H}{\partial z})$)

$$\begin{cases} D_H \Rightarrow H(\Lambda, 1, 1, 0) = H(\Lambda, -1, 1, 0) = H(\Lambda, 0, 1, 0) = 0 = H(\Lambda, 1, 0, 0) = 0 \\ D_S \Rightarrow \nabla H(\Lambda(3, 2, 1)) = \nabla H(\Lambda, 1, 1, 1) = \nabla H(\Lambda, 2, 3, 1) = (0, 0, 0). \end{cases}$$

Solving this linear system of equations in Λ, one gets:

$$\begin{aligned} H(\Lambda, x, y, z) = & \left(\frac{65}{2}\lambda_2 - \frac{8175}{98}\lambda_1\right) z^4 + \left(17\lambda_2 - \frac{1518}{49}\lambda_1\right) yz^3 + \left(-\frac{29}{2}\lambda_2 + \frac{2787}{98}\lambda_1\right) y^2z^2 \\ & + \lambda_2 y^3 z + \left(-97\lambda_2 + \frac{11618}{49}\lambda_1\right) xz^3 + \left(\frac{11}{2}\lambda_2 - \frac{1789}{98}\lambda_1\right) xyz^2 + \left(\frac{9}{2}\lambda_2 - \frac{121}{14}\lambda_1\right) xy^2 z \\ & - \lambda_1 xy^3 + \left(\frac{143}{2}\lambda_2 - \frac{16873}{98}\lambda_1\right) x^2z^2 + \left(-\frac{11}{2}\lambda_2 + \frac{163}{14}\lambda_1\right) x^2yz \\ & + \left(-15\lambda_2 + \frac{254}{7}\lambda_1\right) x^3z + \lambda_1 x^3 y. \end{aligned}$$

Observe that the number of parameters λ_i is 2, and hence $\dim(\mathcal{H}(4, D)) = 1$; see Theorem 3.7. In Fig. 4 one may see 2 different curves in the linear system. The left picture, in Fig. 4, shows a general view in the box $[-300, 300] \times [-300, 300]$. There, one can see the asymptotic behavior established by D_H. The right picture, in Fig. 4, shows a general view in the box $[-3.5, 6] \times [-6, 6]$. There, one can see the 3 double points imposed by D_S. □

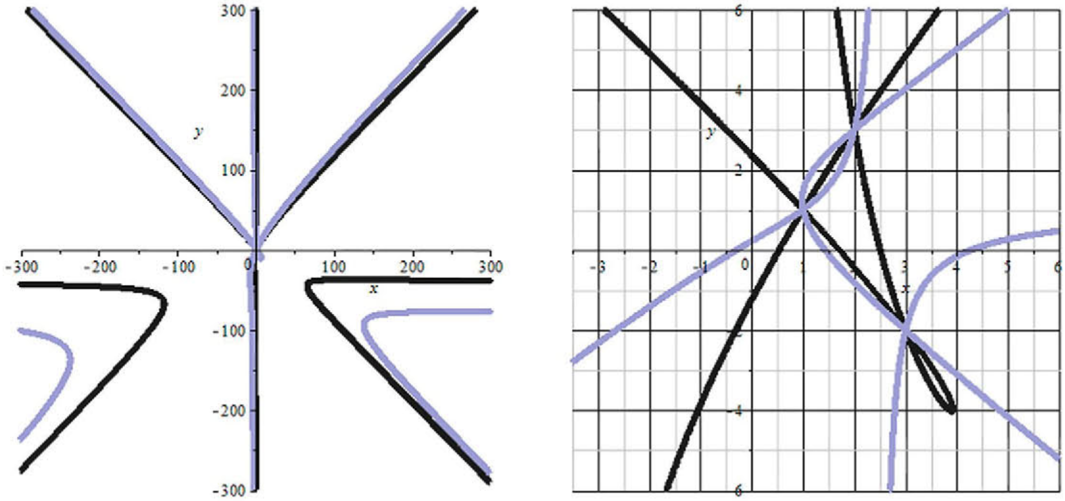

Fig. 4. Two curves in $\mathcal{H}(4, D)$ of Example 3.9. Left: general view in the box $[-300, 300] \times [-300, 300]$. Right: zoom at the singular area in the box $[-3.5, 6] \times [-6, 6]$.

Example 3.10. In Example 3.9, we took $D = D_H + D_S$ with

$$D_H = (1 : 1 : 0) + (-1 : 1 : 0) + (0 : 1 : 0) + (1 : 0 : 0),$$
$$D_S = 2(3 : -2 : 1) + 2(1 : 1 : 1) + 2(2 : 3 : 1),$$

and $\mathcal{H}(4, D)$ was irreducible over $\mathbb{C}$. However, if we replace in D the singular part by $D_S = 2(2 : 2 : 1) + 2(1 : 1 : 1) + 2(2 : 3 : 1)$, then $\mathcal{H}(4, D)$ decomposes as the union of two lines and a system of conics. More precisely, the defining polynomial is

$$4\,(x - 2z)\,(x - y)\left(2x\lambda_2 z + 2xy\lambda_1 - z^2\lambda_2 + 9z^2\lambda_1 - yz\lambda_2 - 13yz\lambda_1 + 2\lambda_1 y^2\right).$$

Obviously the reason, in this example, is that two double points, namely $(2 : 2 : 1)$, $(1 : 1 : 1)$, and one simple point, namely $(1 : 1 : 0)$, are on the line $x - y = 0$, which implies that the line intersects the curve in at least 5 intersections. However, since the curve has degree 4, by Bézout's Theorem (see [36]), the line is a component of the curve. □

The next theorem shows how to analyze the irreducibility of rational Hausdorff divisors.

Theorem 3.11. *Let D be an n-rational Hausdorff divisor. Then, D is irreducible (see Definition 2.6) if and only if the defining polynomial of $\mathcal{H}(n, D)$ is irreducible over $\mathbb{C}$.*

Proof. Let $D = D_H + D_S$. Now, observe that $\mathcal{H}(n, D)$ is a linear subsystem of $\mathcal{H}(n, D_H)$. Now the result follows from Corollary 2.10. □

The bounds in Theorem 3.7 and Corollary 3.8 are equalities in general position, but in some cases are strict inequalities as the following example shows.

Example 3.12. Let $D = D_H + D_S$ be a 5-rational Hausdorff divisor, where $D_S = \sum_{i=1}^{6} 2P_i$ with $P_i = (i^3 : i^2 : 1)$. Theorem 3.7 ensures that $\dim(\mathcal{H}(5, D)) \geq 9 - 12 = -3$; i.e. in general $\mathcal{H}(5, D) = \emptyset$. However, taking e.g. $D_H = \sum(\alpha : 1 : 0)$, with $p(\alpha) = 0$, where

$$p(t) = t^5 - \frac{685587696703}{500} + t^3 - \frac{18732269913}{2000}t^2 + \frac{36724970373}{200}t,$$

it holds that $\dim(\mathcal{H}(5, D)) = 0$, and hence it is not empty. □

The case of non-ordinary singularities

We finish this subsection with a brief description on how one could deal with the case of non-ordinary singularities. We recall that a singular point on a curve is called ordinary if all the tangents to the curve, at the singular point, are different. Otherwise the singularity is called non-ordinary. When the singularity is non-ordinary, the point needs a deeper analysis. This can be done by using, for instance, *blow-ups* at each non-ordinary singularity; see e.g. Sections 3.2 and 3.3 in [36] for an algorithmic version. Here we do not go into the details on blow-ups, but we propose a structure suitable to deal with the case of non-ordinary singularities.

Let $D = \sum_{i=1}^{n} s_i P_i$ be an effective divisor. Then we extend, recursively, the expression of D in the following way. We rewrite D as

$$D = \sum_{i=1}^{n} s_i \cdot (P_i, L_i) \tag{2}$$

where L_i is either the empty list, in which case we understand that P_i is ordinary, or it is a list of the form $L_i = (D^i, \mathcal{T}_i)$ such that

- $D^i = \sum_{j_i=1}^{n_i} s_{i,j_i} \cdot (P_{i,j_i}, \mathcal{T}_{i,j_i})$ is an effective divisor expressed as (2), with all points P_{i,j_i} at the line at infinity (i.e. of the form $(a : b : 0)$), and
- $\mathcal{T}_i$ is a linear projective transformation such that
  1. $\mathcal{T}_i(P_i) = (0 : 0 : 1)$,
  2. $\mathcal{T}_i(P_j) \notin \{(0 : 1 : 0), (1 : 0 : 0)\}$ for $j \neq i$
  3. $\mathcal{T}_i(P_{i_j}) \notin \{(0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0)\}$ for all i_j.

Additionally we impose that for each P_i there are only finitely many nested divisor lists L_i. To clarify the structure, D^i represents the first neighborhood of infinitely near singularities of P_i (geometrically, if we see a curve passing through the divisor, D^i represents the directions of the tangents at P_i with their multiplicities) and $\mathcal{T}_i$ is the suitable projective transformation, used in the blow-up, to move P_i to the origin $(0 : 0 : 1)$ and avoiding that the tangents (i.e. the directions given by P_{i,j_i}) are moved onto the irregular lines of the blow-up, and such that none singularity of D is moved onto the exceptional points of the blow-up.

In this situation, we say that D is n-rational if the formula in Definition 3.1 holds taking also into account the multiplicities in the nested divisors D_{i,j_i}.

Example 3.13. We want to consider a divisor such that it has $(a : 1 : 0)$, where $2a^4 + 1 = 0$, as simple points, and where $(0 : 1 : 1)$ and $(0 : 0 : 1)$ are double points, being $(0 : 0 : 1)$ non-ordinary with a double point in its first neighborhood. Using the terminology introduced above, we consider $D = D_H + D_S$ where

$$D_H = \sum_{2t^4+1=0} ((t : 1 : 0), \emptyset)$$

and

$$D_S = 2 \cdot ((0 : 1 : 1), \emptyset) + 2 \cdot ((0 : 0 : 1), (2 \cdot ((0 : 1 : 0), \emptyset), \{\overline{x} \to (x + y : x - y : z)\})).$$

Since all points in D_H are at infinity and are simple, D is Hausdorff. In addition, $(0 : 1 : 1)$ is an ordinary double point, since it is accompanied by $\emptyset$. Moreover, $(0 : 0 : 1)$ is a double point, and it is non-ordinary because it is accompanied by the pair $(2 \cdot ((1 : 0 : 0), \emptyset), \{\overline{x} \to (x + y : x - y : z)\})$. Furthermore, since the first component of the pair is $((1 : 0 : 0), \emptyset)$, one has that the first neighborhood of $(0 : 0 : 1)$ consists in the ordinary double point $(1 : 0 : 0)$; therefore, we expect the curves to have $y = 0$ as a double tangent at $(0 : 0 : 1)$. Finally, if $\mathcal{T}$ is the $\{\overline{x} \to (x + y : x - y : z)\}$, then $\mathcal{T}(0 : 0 : 1) = (0 : 0 : 1)$, $\mathcal{T}(0 : 1 : 1) = (1 : -1 : 1)$, $\mathcal{T}(t : 1 : 0) = (t + 1 : t - 1 : 0) \notin \{(0 : 1 : 0), (1 : 0 : 0)\}$ and $\mathcal{T}(1 : 0 : 0) = (1 : 1 : 0) \notin \{(0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0)\}$. Moreover, D is 4-rational since we have three double points, namely $(0 : 1 : 0)$, $(0 : 0 : 1)$ and $(1 : 0 : 0)$ infinitely near $(0 : 0 : 1)$.

Let us see, now, how to compute $\mathcal{H}(4, D)$. We consider a general homogeneous polynomial H in $\{x, y, z\}$ with undetermined coefficients Λ. The condition on D_H is equivalent to require that the quotient w.r.t. t of $H(\Lambda, t, 1, 0)$, when divided by $2t^2 + 1$, is identically zero. This provides 4 linear equations in Λ. The condition on $(0 : 1 : 1)$ is equivalent to asking for the gradient of H, ∇H, to vanish at $(0 : 1 : 1)$. This provides 3 linear equations in Λ. Let us now work with the new part: the non-ordinary double point $(0 : 0 : 1)$.

1. First we ask $(0 : 0 : 1)$ to be a double point; that is, $\nabla H(0, 0, 1) = (0, 0, 0)$. This provides 3 linear equations in Λ.
2. Second, we require the curve to have $y = 0$ as a double tangent at $(0 : 0 : 1)$. This is equivalent to asking for the coefficient of H, w.r.t. z^2, to be equal to μy^2, where μ is a new parameter. This implies 3 new linear equations in Λ.
3. We apply the linear transformation $\mathcal{T}$ to H to afterwards apply the Cremona transformation $\{\overline{x} \to (yz : xz : xy)\}$. So we compute $M(x, y, z) = H(yz + xz, yz - xz, xy)$. Finally, after crossing out, from M, the factors of the form x^i, y^j and z^k, we get $N(x, y, z)$. Now we require that $N(0 : 1 : 0) = 0$. This implies 1 linear equation in Λ.

Solving the set of all linear equations, we get that the defining polynomial of $\mathcal{H}(4, D)$ is $H = 2\mu x^4 + \mu y^4 - 2\mu y^3 z + \mu y^2 z^2 + x^2 yz\lambda_{11}$. Observe that $\dim(\mathcal{H}(4, D)) = 2$ and compare to Theorem 3.7. Now, every irreducible curve in $\mathcal{H}(4, D)$ is compact (note that all points in D_H are complex), is rational, has $(0 : 1 : 1)$ and $(0 : 0 : 1)$ as double points, and the tangent to the curve at $(0 : 0 : 1)$ is $y = 0$ with multiplicity 2. In Fig. 5, one can see two elements in $\mathcal{H}(4, D)$.

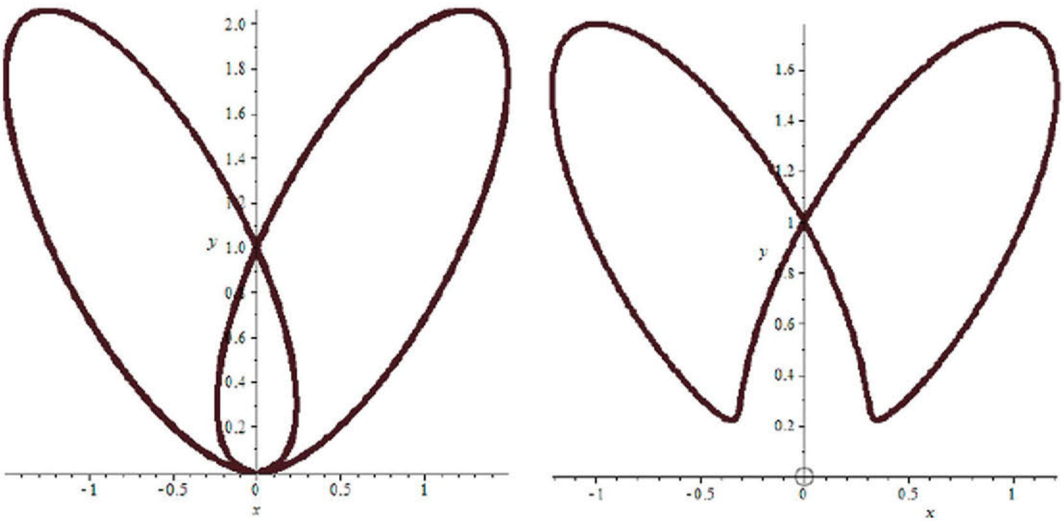

Fig. 5. Two curves in Example 3.13 Left: the tacnote curve. Right: another curve in $\mathcal{H}(4, D)$ with an isolated double singularity at the origin.

3.2. *Parametrization of rational Hausdorff linear systems*

Let $\mathbb{K}$ be a subfield of $\mathbb{C}$, and let $D = D_H + D_S$ be a $\mathbb{K}$-definable n-rational Hausdorff divisor. Our goal in this section is to parametrize the curve $\mathsf{Curve}(\mathcal{H}(n, D))$ associated to D; that is the curve defined, over the algebraic closure of $\mathbb{C}(\Lambda)$, by the defining polynomial $H(\Lambda, x, y, z)$ of the rational Hausdorff linear space $\mathcal{H}(n, D)$ (see Definition 2.1). Recall that, by Theorem 3.5, if D is irreducible, then $\mathsf{Curve}(\mathcal{H}(n, D))$ is rational.

Thus, throughout this section we assume that D is irreducible (see Definition 2.6 and Theorem 3.11) which, in particular, implies that $\mathcal{H}(n, D)$ is not empty (see also Theorem 3.7 and Corollary 3.8). Moreover, let $H(\Lambda, x, y, z) \in \mathbb{K}[\Lambda][x, y, z]$, where Λ is a set of parameters, be the defining polynomial of $\mathcal{H}(n, D)$; observe that the $\mathbb{K}$-definability of D implies that H is a polynomial over $\mathbb{K}$.

But before going into details, let us recall, at least intuitively, how the parametrization algorithms, based on adjoint curves, work. Since, we will be dealing only with ordinary singularities we simplify the exposition to that case; for further details, see [36]. Say that $\mathcal{C}$ is a rational projective curve of degree k. The simplest case is when $\mathcal{C}$ is monomial; let P be the $(k-1)$ fold-point of $\mathcal{C}$. In this situation, the intersection of $\mathcal{C}$ with $\mathsf{Curve}(\mathcal{H}(1, P))$ consists in P and an additional point that depends rationally on a parameter. This last point is indeed a parametrization of $\mathcal{C}$. This method is called parametrization by lines. In general, let $\{P_1, \ldots, P_s\}$ be the singularities of $\mathcal{C}$, then an adjoint curve to $\mathcal{C}$ of degree ℓ (in general, $\ell \geq k-2$) is any curve in the linear system of curves

$$\mathcal{H}\left(\ell, \sum_{i=1}^{s}(\mathrm{mult}(\mathcal{C}, P_i) - 1)P_i\right).$$

Let $\mathcal{A}_\ell(\mathcal{C})$ denote the linear system above, that is the linear system of all adjoints to $\mathcal{C}$ of degree ℓ. Because of the genus formula and the dimension of $\mathcal{A}_\ell(\mathcal{C})$ it holds that taking a finite set of simple points $\{Q_1, \ldots, Q_r\}$ of $\mathcal{C}$, for a suitable r, and considering $\mathcal{H}^* := \mathcal{A}_\ell(\mathcal{C}) \cap \mathcal{H}(\ell, Q_1 + \cdots + Q_r)$ it holds that the intersection of $\mathcal{C}$ with $\mathsf{Curve}(\mathcal{H}^*)$ consists in $\{P_1, \ldots, P_s\} \cup \{Q_1, \ldots, Q_r\}$ and an additional point that depends rationally on a parameter. This last point is indeed a parametrization of $\mathcal{C}$. Let us assume that the homogeneous form defining $\mathcal{C}$ has coefficients in $\mathbb{K}$. Then an important property, of these type of algorithms, is that the coefficients of the parametrization (field of parametrization) are in $\mathbb{K}$ (if $\mathcal{C}$ was parametrized by lines) or in the smallest field containing $\mathbb{K}$ and the coefficients of the chosen points $\{Q_1, \ldots, Q_s\}$.

As we said, our goal is to parametrize $\mathsf{Curve}(\mathcal{H}(n, D))$, but sometimes, we will also parametrize the curve $\mathsf{Curve}(\overline{\mathcal{H}})$ associated to a non-empty linear subsystem $\overline{\mathcal{H}}$ of $\mathcal{H}(n, D)$. Applying the well-known parametrization algorithms, since the coefficients of the input curve are in $\mathbb{K}(\Lambda)$, one derives a rational parametrization of $\mathsf{Curve}(\mathcal{H}(n, D))$ over the algebraic closure of $\mathbb{C}(\Lambda)$. The challenge is to parametrize $\mathsf{Curve}(\mathcal{H}(n, D))$ over the smallest possible field extension of $\mathbb{K}(\Lambda)$. We start observing that, as a consequence of Hilbert–Hurwitz's Theorem (see Theorem 5.8 in [36]) and Tsen's Theorem (Corollary 4 in [42, Vol. I. p. 73]), every irreducible linear subsystem of dimension 0 or 1 of $\mathcal{H}(n, D)$ is parametrizable over $\mathbb{C}(\overline{\Lambda})$, where $\overline{\Lambda}$ are the parameters involved in the definition of the subsystem. Nevertheless, as a consequence of the Hausdorff divisor, we can improve this statement (note that no hypothesis on the dimension is required). We recall that proper means that the parametrization defines a 1:1 map from a non-empty Zariski open subset of the parameter space and the curve.

Theorem 3.14 (*General Parametrization Theorem*). *There exists a rational proper parametrization of* $\mathsf{Curve}(\mathcal{H}(n, D))$ *with coefficients in* $\mathbb{L}(\Lambda)$, *where* $\mathbb{L}$ *is a finite algebraic extension of* $\mathbb{K}$ *of degree at most* n. *Furthermore, the degree of the extension is the lowest degree of the nontrivial irreducible factors, in* $\mathbb{K}[\Lambda][x, y]$ *of* $H(\Lambda, x, y, 0)$.

Proof. Since D_H is Hausdorff, and $\deg(D_H) = \deg(\text{Curve}(\mathcal{H}(n, D)))$, by Bézout's Theorem it holds that all points of Curve $(\mathcal{H}(n, D))$ on the line $z = 0$ are simple. Moreover, these points are over $\mathbb{C}$. Furthermore, since D_H is $\mathbb{K}$-definable, these points at infinity form a $\mathbb{K}$-conjugate family of points that can be decomposed as a union of families, each defined by a factor of $H(\Lambda, x, y, 0)$ in $\mathbb{K}[\Lambda][x, y]$; say that k is the lowest degree of these factors. On the other hand, since D_S is $\mathbb{K}$-definable, one has that the linear system of n-degree adjoint curves to Curve$(\mathcal{H}(n, D))$ can be defined over $\mathbb{K}$ (see Theorem 4.66 in [36]). Therefore, using the parametrization algorithm by n-degree adjoint curves (see Section 4.8 in [36]) and taking the simple point in one of the families of cardinality k, one deduces that Curve$(\mathcal{H}(n, D))$ can be properly parametrized over $\mathbb{L}(\Lambda)$, where $\mathbb{L}$ is a finite algebraic extension of $\mathbb{K}$ of degree k. □

From the previous proof one can derive an algorithm to parametrize Curve$(\mathcal{H}(n, D))$ over $\mathbb{L}(\Lambda)$. Indeed, the extension $\mathbb{L}$ is the extension needed to express the simple points in D_H used in the parametrization algorithm. In the following we analyze how to decrease the degree of the extension in some special cases.

Corollary 3.15. *If one of the points in D_H has coordinates over $\mathbb{K}$, there exists a rational proper parametrization of* Curve $(\mathcal{H}(n, D))$ *with coefficients in $\mathbb{K}(\Lambda)$.*

Example 3.16. Let D be the 4-rational divisor in Example 3.9. Since D_H has points in $\mathbb{Q}$, Corollary 3.15 ensures that Curve$(\mathcal{H}(4, D))$ can be parametrized over $\mathbb{Q}(\Lambda)$. Indeed, we can consider 2-degree adjoints and we use the simple point $(1 : 0 : 0)$. That is, we consider the divisor $D^* = (3 : -2 : 1) + (1 : 1 : 1) + (2 : 3 : 1) + (1 : 0 : 0)$, and $\mathcal{H}(2, D^*)$. Note that $\dim(\mathcal{H}(2, D^*)) = 1$. In this situation, let $F(\Lambda, x, y, z)$ be the defining polynomial of $\mathcal{H}(4, D)$ and let $G(t, x, y, z)$ be the defining polynomial of $\mathcal{H}(2, D^*)$ (recall that $\dim(\mathcal{H}(2, D^*)) = 1$). Then, the primitive part w.r.t. t (we denote it by R_1) of the resultant of $F(\Lambda, x, y, 1)$ and $G(t, x, y, 1)$ w.r.t. y is a linear polynomial in x with coefficients depending on Λ and t (see e.g. [40] for the notions of primitive part and resultant). Similarly, the primitive part w.r.t. t (we denote it by R_2) of the resultant of $F(\Lambda, x, y, 1)$ and $G(t, x, y, 1)$ w.r.t. x is a linear polynomial in y with coefficients depending on Λ and t. More precisely, one gets

$$R_1(\Lambda, t, x) = A_2(\Lambda, t)x - A_1(\Lambda, t), \qquad R_2(\Lambda, t, y) = A_4(\Lambda, t)y - A_3(\Lambda, t)$$

where

$$\begin{aligned} A_1 &= -238\lambda_1 t^3\lambda_2 + 2240\lambda_2 t^2\lambda_1 + 98{\lambda_1}^2 t^3 - 2787{\lambda_2}^2 t^2 + 1470{\lambda_1}^2 - 6986\lambda_1\lambda_2 \\ &\quad - 539{\lambda_1}^2 t + 8328{\lambda_2}^2 + 1792\lambda_1\lambda_2 t - 441{\lambda_1}^2 t^2 - 1209{\lambda_2}^2 t, \\ A_2 &= 14\lambda_2\left(2\lambda_2 - 2\lambda_2 t^2 - 17\lambda_2 t^3 + 17\lambda_2 t - 7\lambda_1 t + 7\lambda_1 t^3\right), \\ A_3 &= 486\lambda_2 + 77\lambda_1 t + 34\lambda_2 t^3 + 63\lambda_1 t^2 - 145\lambda_2 t - 147\lambda_2 t^2 - 14\lambda_1 t^3 - 210\lambda_1, \\ A_4 &= 14\lambda_2\left(-1 + t^2\right). \end{aligned}$$

Solving the linear equations $\{R_1 = 0, R_2 = 0\}$ in $\{x, y\}$ one gets the parametrization

$$\left(\frac{A_1(\Lambda, t)}{A_2(\Lambda, t)}, \frac{A_3(\Lambda, t)}{A_4(\Lambda, t)}, 1\right).$$

Example 3.17. Let us consider the 4-degree $\mathbb{Q}$-definable rational Hausdorff divisor

$$D = D_H + D_S \quad \text{where } D_H = \sum_{t^4-4=0} (1 : t : 0), \ D_S = 2\sum_{t^3+1=0} (t : t^2 : 1).$$

From D one sees that the curves in $\mathcal{H}(4, D)$ have two real asymptotes in the directions $(1, \pm\sqrt{2})$ as well as three double points, one of them real; namely $(-1 : 1 : 1)$ (see Fig. 6). The linear system $\mathcal{H}(4, D)$ associated to D is given by

$$\begin{aligned} H(\Lambda, x, y, z) &= -\lambda_1 z^4 + \lambda_2 y^2 z^2 - \lambda_1 y^3 z - 4a_{4,0} y^4 - 3\lambda_1 xyz^2 - 8\lambda_2 xy^2 z - 4\lambda_2 x^2 z^2 \\ &\quad - 2\lambda_2 x^2 yz + \lambda_1 x^3 z + \lambda_2 x^4. \end{aligned}$$

Observe that $\dim(\mathcal{H}(4, D)) = 1$ and compare to Theorem 3.7. In addition, we observe that the Hausdorff divisor can be expressed by conjugate families as

$$D_H = \sum_{t^2-2=0} (1 : t : 0) + \sum_{t^2+2=0} (1 : t : 0).$$

Corollary 3.15 ensures that Curve$(\mathcal{H}(4, D))$ can be parametrized over $\mathbb{Q}(\sqrt{2})(\Lambda)$. Indeed, if we take 2-degree adjoints and we use the simple point $(1 : \sqrt{2} : 0)$, reasoning as in Example 3.16, we get the parametrization

$$\left(\frac{A_1(\Lambda, t)}{B(\Lambda, t)}, \frac{A_2(\Lambda, t)}{B(\Lambda, t)}, 1\right)$$

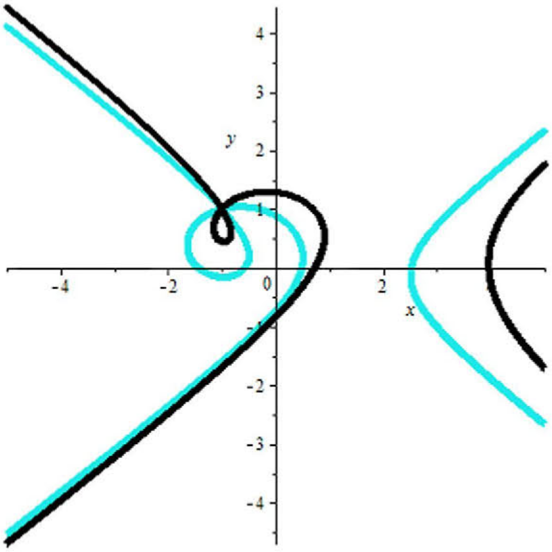

Fig. 6. Two curves of $\mathcal{H}(4, D)$ in Example 3.17.

where

$$\begin{aligned} A_1 &= \frac{1}{14}\left(-4+\sqrt{2}\right)\left(\sqrt{2}t^4\lambda_1{}^2 + 33\lambda_1{}^2t\sqrt{2} + 16t\lambda_2\lambda_1 - 4\lambda_2{}^2t^3\sqrt{2} + 12\lambda_2t^3\lambda_1\right. \\ &\quad + 16t^3\sqrt{2}\lambda_1{}^2 + 16t^4\lambda_1\lambda_2 + 12\lambda_1{}^2t^2\sqrt{2} + 32t^2\lambda_2\lambda_1 + 20\lambda_1{}^2t + 14t^4\lambda_2{}^2 \\ &\quad - 16t^3\lambda_2{}^2 + 8t^3\lambda_1{}^2 + 48\lambda_1{}^2t^2 + 4t^4\lambda_1{}^2 + 4\sqrt{2}t^4\lambda_1\lambda_2 + 4\lambda_2\lambda_1t\sqrt{2} \\ &\quad \left. + 31t^3\lambda_2\lambda_1\sqrt{2} + 8\lambda_2t^2\sqrt{2}\lambda_1 + 18\lambda_1{}^2 + 8\sqrt{2}\lambda_1{}^2\right), \\ A_2 &= -\frac{1}{7}\left(1+2\sqrt{2}\right)\left(-16\lambda_2{}^2t^3\sqrt{2} + 3\lambda_1{}^2t^2\sqrt{2} + 8t\lambda_2\lambda_1 - 12\lambda_1{}^2t\sqrt{2} + 10\lambda_2t^3\lambda_1\right. \\ &\quad - 4t^4\lambda_1\lambda_2 - 50t^2\lambda_2\lambda_1 + 6\lambda_1{}^2t - 7t^4\lambda_2{}^2 + 8t^3\lambda_2{}^2 - 2t^3\sqrt{2}\lambda_1{}^2 - 12\lambda_1{}^2t^2 \\ &\quad \left. + \sqrt{2}t^4\lambda_1\lambda_2 - 16\lambda_2\lambda_1t\sqrt{2} + t^3\lambda_1{}^2 + 16\lambda_2t^2\sqrt{2}\lambda_1 - 7\lambda_1{}^2 - 20t^3\lambda_2\lambda_1\sqrt{2}\right), \end{aligned}$$

$$B = \lambda_2 t\left(t^3\lambda_1 + 12\lambda_1 t + 4\lambda_2 t^3 + 16t\lambda_2 + 4t^2\lambda_1\sqrt{2} + 8\lambda_1\sqrt{2} + 8\lambda_2 t^2\sqrt{2}\right).$$

Corollary 3.18. *If* $\dim(\mathcal{H}(n, D)) > 0$, *for every* $P \in \mathbb{P}^2(\mathbb{K})$ *such that* $\overline{\mathcal{H}} := \mathcal{H}(n, D) \cap \mathcal{H}(n, P)$ *is irreducible, then* $\mathrm{Curve}(\overline{\mathcal{H}})$ *can be rationally and properly parametrized over* $\mathbb{K}(\Lambda)$.

Proof. It follows by using P in the parametrization algorithm. □

Example 3.19. Let D be as in Example 3.17. Since $\dim(\mathcal{H}(4, D)) = 1$ we can apply Corollary 3.18 and get a better parametrization than the one given in Example 3.17, namely, with coefficients in $\mathbb{Q}$ instead of in $\mathbb{Q}(\sqrt{2})$. The idea is that we use the free parameter defining $\mathcal{H}(4, D)$ (recall that $\dim(\mathcal{H}(4, D)) = 1$) to get a subsystem passing through a fixed simple point. We do it in general, taking a generic affine point. We take a generic affine point $P := (a : b : 1) \in \mathbb{P}^2(\mathbb{C})$ and we consider $\overline{\mathcal{H}} = \mathcal{H}(4, D) \cap \mathcal{H}(4, P)$; this implies to solve the linear equation in Λ given by $H(\Lambda, a, b, 1) = 0$, where H is the defining polynomial $\mathcal{H}(4, D)$ (see Example 3.17). In order to avoid reducibility, computations show that P has to be taken not satisfying the equation

$$(a - 1 - b)(a^2 + a + ab - b + 1 + b^2) = 0.$$

The defining polynomial of $\overline{\mathcal{H}}$ is

$$\begin{aligned} \overline{H}(x, y, z) &= -24xyz^2ab^2 - 6xyz^2a^2b + 24xy^2zab + 6x^2yzab + 4y^4 - x^4 - 12xyz^2a^2 + 3xyz^2a^4 \\ &\quad + 8xy^2zb^3 - 8xy^2za^3 + 12x^2z^2ab + 2x^2yzb^3 - 2x^2yza^3 + 8x^3zab^2 + 2x^3za^2b - 3y^2z^2ab \\ &\quad - 8y^3zab^2 - 2y^3za^2b + 3xyz^2b^2 - 12xyz^2b^4 - 4y^3zb^4 - x^3za^4 + 4x^2z^2b^3 + y^3za^4 - y^2z^2b^3 \\ &\quad - 4x^2z^2a^3 + 12y^4ab - x^3zb^2 - 2z^4a^2b + 4x^3zb^4 + y^3zb^2 + y^2z^2a^3 + 8xy^2z - 8z^4ab^2 \\ &\quad - 4y^3za^2 - 3x^4ab + 2x^2yz + 4x^3za^2 - y^2z^2 + 4x^2z^2 + z^4b^2 - 4z^4b^4 - 4z^4a^2 + z^4a^4 \\ &\quad + 4y^4b^3 - 4y^4a^3 - x^4b^3 + x^4a^3. \end{aligned}$$

In this situation, we consider the 1-dimensional linear system of conics $\mathcal{H}^* = \mathcal{H}(2, \sum_{h^3+1=0}(h : h^2 : 1)) \cap \mathcal{H}(2, P)$. The defining polynomial of $\mathcal{H}^*$ is

$$\begin{aligned} H^*(t, x, y, z) &= z^2b^2 + z^2a - tyzb^2 - tyza - y^2 + y^2tb - y^2ab - y^2ta^2 - xz + xztb \\ &\quad - xzab - xzta^2 + xyb^2 + xya + tx^2b^2 + tx^2a. \end{aligned}$$

Then, the intersection of $\overline{\mathcal{H}}$ and $\mathcal{H}^*$ provides a parametrization of $\mathrm{Curve}(\overline{\mathcal{H}})$ with coefficients in $\mathbb{Q}(a, b)$; we do not show the output here because it is too large. Instead, we illustrate it with particular values of P, for instance $P = (1 : 1 : 1)$. We get

$$\left(\frac{1024 + 1024t + 960t^4 + 4096t^2 + 4352t^3}{-4\left(16 - 16t - 4t^2\right)\left(16 + 16t + 12t^2\right)}, \frac{1024 + 2048t - 192t^4 + 3584t^2 + 512t^3}{4\left(16 - 16t - 4t^2\right)\left(16 + 16t + 12t^2\right)}, 1\right).$$

Similarly, for $P = (0 : 1 : 1)$ we get

$$\left(-\frac{1}{2}\frac{18 - 15t + 6t^2 + 74t^3 + 21t^4}{\left(-t^2 - 6t + 9\right)\left(3t^2 + 2t + 1\right)}, \frac{1}{2}\frac{-3t - 6t^4 + 75t^2 + 23t^3}{\left(-t^2 - 6t + 9\right)\left(3t^2 + 2t + 1\right)}, 1\right).$$

Both parametrizations have coefficients in $\mathbb{Q}$. □

Let us assume that D is monomial, then one can parametrize $\mathrm{Curve}(\mathcal{H}(n, D))$ by lines (see Section 4.6 in [36]). In addition, since D is $\mathbb{K}$ definable, then the field of parametrization is $\mathbb{K}(\Lambda)$. Therefore, one has the following theorem.

Theorem 3.20. *Let D be monomial, then there exists a rational proper parametrization of $\mathrm{Curve}(\mathcal{H}(n, D))$ with coefficients in $\mathbb{K}(\Lambda)$.*

Example 3.21. We consider the divisor

$$D = \sum_{t^4+1=0} (t : 1 : 0) + 3(0 : 0 : 1).$$

D is a $\mathbb{Q}$-definable 4-monomial Hausdorff divisor. So, by Corollary 3.8, $\dim(\mathcal{H}(4, D)) = 4$. Indeed, the defining polynomial of $\mathcal{H}(4, D)$ is

$$H = \lambda_1 y^3 z + \lambda_2 y^4 + \lambda_3 x y^2 z + \lambda_5 x^2 y z + \lambda_4 x^3 z + \lambda_2 x^4.$$

We observe that H is irreducible over $\mathbb{C}$, and hence D is irreducible (see Theorem 3.11). Since all points in the Hausdorff divisor are complex, we know that the curves in $\mathcal{H}(4, D)$ are compact and, from the rational divisor, with a triple point at the origin (see Fig. 7). Now, parametrizing with the pencil of lines $ty + x = 0$ one gets the parametrization of the linear system

$$\mathcal{P}(t) = \left(\frac{-t\left(t\lambda_3 - \lambda_1 + t^3\lambda_4 - t^2\lambda_5\right)}{\lambda_2\left(t^4 + 1\right)}, \frac{t\lambda_3 - \lambda_1 + t^3\lambda_4 - t^2 a_{2,1}}{\lambda_2\left(t^4 + 1\right)}, 1\right).$$

Theorem 3.22. *Let D_S have at least a triple point over $\mathbb{K}$, then there exists a rational proper parametrization of $\mathrm{Curve}(\mathcal{H}(n, D))$ with coefficients in $\mathbb{K}(\Lambda)$.*

Proof. Using the triple point one can generate families of $(n - 3)$ conjugate points over $\mathbb{K}(\Lambda)$ (see Section 3.3 in [36]) to afterwards parametrize with $(n - 2)$-degree adjoint curves (see Section 4.7 in [36]). □

4. Application to the approximate parametrization problem: Hausdorff curves

Given a non-rational irreducible curve, the approximate parametrization problem consists in providing a rational curve being at close Hausdorff distance of the input curve; see Section 1.2 for further details. In this section we show, as a sample of application of the ideas developed, that every Hausdorff curve (see definition below) can always be parametrized approximately.

Definition 4.1. We say that an affine plane algebraic curve $\mathcal{C}$ is a Hausdorff curve if $\mathrm{card}(\mathcal{C}^\infty) = \deg(\mathcal{C})$; recall that $\mathcal{C}^\infty$ denotes the points at infinity of $\mathcal{C}$.

Example 4.2. Observe that all lines are Hausdorff and the only conics that are not Hausdorff are the parabolas. For degree 3 or higher the number of possibilities increases.

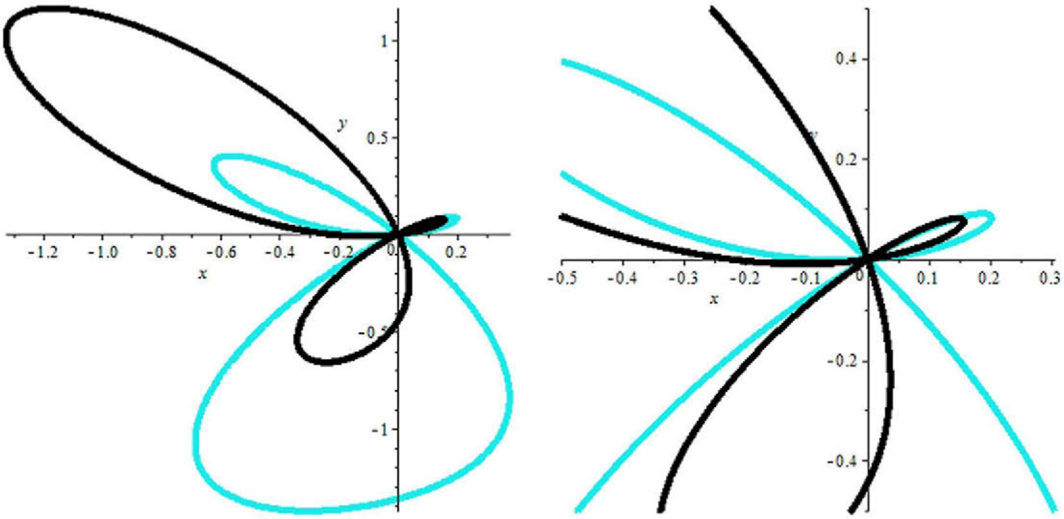

Fig. 7. Two curves of $\mathcal{H}(4, D)$ in Example 3.21. Left: general view. Right: zoom on the box $[-0.5, 0.3] \times [-0.5, 0.5]$ where one can see the origin as triple point.

Remark 4.3. Observe that, if $\mathcal{C}$ is a $\mathbb{K}$-definable Hausdorff curve of degree n, then

$$D = \sum_{P \in \mathcal{C}^{\infty}} P$$

is an n-degree $\mathbb{K}$-definable Hausdorff divisor. We call D the Hausdorff divisor associated to $\mathcal{C}$.

The following theorem states the main property of Hausdorff curves.

Theorem 4.4. *Let $\mathcal{C}$ be a real irreducible affine Hausdorff curve of degree n, and let $D = \sum_{i=1}^{n}(a_i : b_i : 0)$ be its associated Hausdorff divisor. Then, for every point $P = (a : b : 1) \in \mathbb{P}^2(\mathbb{C})$, such that $ab_i - ba_i \neq 0$ with $i = 1, \ldots, n$, $D + (n-1)P$ is an irreducible Hausdorff monomial divisor.*

Proof. Let $\overline{D} = D + (n-1)P$. Taking $\overline{D}_H = D$ and $\overline{D}_S = (n-1)P$, one has that $\overline{D}$ is Hausdorff and monomial. In order to prove that $\overline{D}$ is irreducible, by Corollary 2.10, we prove that the defining polynomial $\overline{H}(\Lambda, x, y, z)$ of $\mathcal{H}(n, \overline{D})$ is irreducible over $\mathbb{C}(\Lambda)$. Moreover, since P has coefficients in $\mathbb{C}$, we can consider w.l.o.g. that $P = (0 : 0 : 1)$; otherwise one performs a suitable linear change over $\mathbb{C}$. In this situation, the proof is analogous to the proof of Lemma 2.8. □

The next results follow from Theorems 4.4, 3.20 and 2.11, and Corollary 3.8.

Corollary 4.5. *Let $\mathbb{K}$ be a subfield of $\mathbb{C}$, let $\mathcal{C}$ be a real irreducible affine $\mathbb{K}$-definable Hausdorff curve of degree n. Then, there exist infinitely many real monomial plane curves $\mathcal{D}$, parametrizable over $\mathbb{K}$, such that $\mathrm{d}(\mathcal{C} \cap \mathbb{R}^2, \mathcal{D} \cap \mathbb{R}^2) < \infty$. Furthermore, for any fixed point P, chosen as in Theorem 4.4, the dimension of the linear system of n-degree monomial curves, having P as singular point, is n.*

Corollary 4.6 (*Approximate Parametrization of Hausdorff Curves*). *Let $\mathcal{C}$ be a real irreducible affine Hausdorff curve of degree n. Then, for every P chosen as in Theorem 4.4, there exists an n-dimensional linear system where all real irreducible curves are solutions of the approximate parametrization problem applied to $\mathcal{C}$.*

From the previous result, one may proceed as follows. Let us say that we are given a real irreducible affine Hausdorff curve $\mathcal{C}$ of degree n, and we want to provide a rational curve $\mathcal{D}$, at finite Hausdorff distance of $\mathcal{C}$, passing through a fixed affine point P. We may assume that P satisfies the conditions in Theorem 4.4, otherwise we apply a small perturbation to P. In this situation, one computes $\overline{\mathcal{H}} = \mathcal{H}(n, D + (n-1)P)$, where D is the Hausdorff divisor associated to $\mathcal{C}$. We know that almost all curves in $\overline{\mathcal{H}}$ are irreducible, and hence rational. Moreover, we know that $\dim(\overline{\mathcal{H}}) = n$. That is, we still have n degrees of freedom to chose a suitable (under the requirements stated by the user) rational curve to our particular problem. For instance, one may look for a rational curve in $\overline{\mathcal{H}}$ under the criterium of minimizing the Hausdorff distance, or reducing the length of the coefficients in the parametrization, or passing through the ramification points of $\mathcal{C}$, or having particular tangents at particular points, etc.

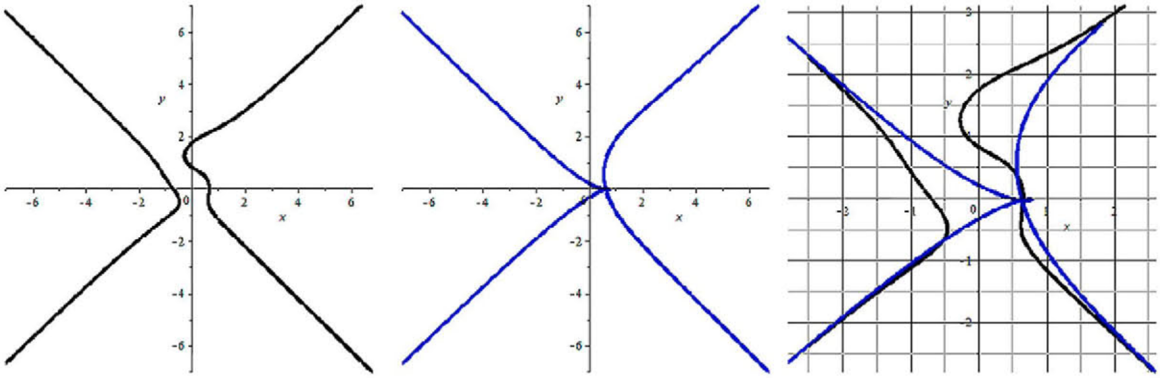

Fig. 8. Left: plot of $\mathcal{C}$ in Example 4.7. Center: plot of $\mathcal{D}$ in Example 4.7. Left: zoom of $\mathcal{C}$ and $\mathcal{D}$ in the box $[-3, 3] \times [-3, 3]$.

We finish this section, illustrating these ideas by an example.

Example 4.7. We consider the affine curve $\mathcal{C}$ defined by

$$4 + 2y - 5y^2 - 9y^3 + 6y^4 + x - 7xy - 5xy^2 - 6x^2 + 6x^2y - 3x^3 - 6x^4.$$

$\mathcal{C}$ is real, irreducible and has degree 4. Moreover, $\mathcal{C}^\infty = \{(1 : \pm 1 : 0), (1 : \pm i : 0)\}$. Therefore, $\mathcal{C}$ is Hausdorff and its associated Hausdorff divisor is

$$D = \sum_{t^4=1} (1 : t : 0).$$

From D we know that $\mathcal{C}$ has two real asymptotes. On the other hand, $\mathcal{C}$ has genus 3. Thus, since $\deg(\mathcal{C}) = 4$, $\mathcal{C}$ is smooth (see Fig. 8, left). We take a point, for instance $P = (41/64 : -1/32 : 1)$, satisfying the conditions of Theorem 4.4; P has been taken as an approximation of a ramification point of $\mathcal{C}$. Then, $\overline{D} = D + 3P$ is an irreducible Hausdorff monomial divisor. The associated linear system $\mathcal{H}(4, \overline{D})$ is defined by

$$\begin{aligned} H = {} & \lambda_4 x^3 z + 2825745/524288 yz^3\lambda_4 + 1024xy^2z\lambda_2 + 1312xy^2z\lambda_3 + 1681/4y^2z^2\lambda_3 \\ & - \lambda_4 y^4 + 13448y^3z\lambda_3 + 7236657/512y^3z\lambda_4 + \lambda_4 x^4 + 32x^2yz\lambda_3 + \lambda_3 x^2z^2 \\ & - 68921/2048xyz^2\lambda_4 + 1312y^2z^2\lambda_2 + \lambda_2 xz^3 + 41xyz^2\lambda_3 + 32768y^3z\lambda_1 \\ & + 96yz^3\lambda_1 + 3072y^2z^2\lambda_1 + 64xyz^2\lambda_2 + 8979/64x^2yz\lambda_4 + \lambda_1 z^4 + 41/2yz^3\lambda_2 \\ & - 2825809/16384y^2z^2\lambda_4 + 149609/64xy^2z\lambda_4 + 20992y^3z\lambda_2. \end{aligned}$$

As expected, $\dim(\mathcal{H}(4, \overline{D})) = 4$. Moreover, for every $\Lambda_0 \in \mathbb{C}^5$, such that $H(\Lambda_0, x, y, z)$ is irreducible over $\mathbb{C}$, we get a monomial curve. Furthermore, the affine curve $H(\Lambda_0, x, y, 1)$ is monomial and is at finite distance of $\mathcal{C}$. Since we have 4 degrees of freedom, we choose the curve such that it passes through 4 points of $\mathcal{C}$. We intersect $\mathcal{C}$ with the lines $y = \pm 3$ to get

$$Q_1 = \left(\frac{89}{32} : -3 : 1\right), \qquad Q_2 = \left(-\frac{101}{32} : -3 : 1\right), \qquad Q_3 = \left(\frac{65}{32} : 3 : 1\right), \qquad Q_4 = \left(-\frac{103}{32} : 3 : 1\right).$$

We consider $\overline{\overline{\mathcal{H}}} = \mathcal{H}(4, D + 3P + Q_1 + Q_2 + Q_3 + Q_4)$. We note that $\dim(\overline{\overline{\mathcal{H}}}) = 0$ and consists in the curve $\mathcal{D}$ defined by (in Fig. 8, right, one can see how both curves $\mathcal{C}$ and $\mathcal{D}$ intersect at Q_1, Q_2, Q_3, Q_4)

$$\begin{aligned} G = {} & -1118978050438561737380 8yz^3 - 641774463845079068940 80y^2z^2 \\ & + 253289290451266902712 32y^3z - 683156633511819645747 20x^3z \\ & + 694464732023697206958 08x^2z^2 - 309494726477141109136 96xz^3 \\ & - 248972113943285307801 60y^4 + 248972113943285307801 60x^4 \\ & + 286774787435937948272 64xyz^2 + 104113819442735106875 392xy^2z \\ & - 173036995343788102615 04x^2yz + 509464984368695582498 5z^4. \end{aligned}$$

$\mathcal{D}$ is rational and can be parametrized by lines through P as

$$\left(\frac{1}{94975324227632640}\frac{A_1(t)}{B(t)}, -\frac{1}{47487662113816320}\frac{A_2(t)}{B(t)}, 1\right)$$

where

$$\begin{aligned} A_1 &= 208880643591165188824385 + 1309845452973236446822400t^4 + 315234830455113833655603 2t^2 \\ &\quad + 132660992099263192594377 6t + 332187157417516077445939 2t^3, \\ A_2 &= 309494726477141109136 96t + 683156633511819645747 20t^3 + 694464732023697206958 08t^2 \\ &\quad + 509464984368695582498 5 + 248972113943285307801 60t^4, \\ B(t) &= 2825745 + 41312256t^2 + 42991616t^3 + 16777216t^4 + 17643776t. \end{aligned}$$

Furthermore, $\mathcal{D}$ and $\mathcal{C}$ are at finite distance of $\mathcal{C}$ (see Fig. 8). □.

5. Conclusions and future directions of research

In order to deal with the approximate parametrization problem, one needs to ensure that the Hausdorff distance between input and output curves is finite, and that the output curve is rational. For the finite Hausdorff distance, we introduce the new concept of Hausdorff divisor, and for the rationality we use the notion of rational divisors. Then all real curves associated to a Hausdorff divisor D_H, whose degree is the degree of D_H, are at finite Hausdorff distance among then; we call Hausdorff curves to this type of curves. To deal with the second issue, the rationality, we add to the Hausdorff divisor D_H a rational divisor D_S. Then, the set of all curves associated to the new divisor $D_H + D_S$, with degree again the degree of D_H, is a linear projective space $\mathcal{H}$, and all real irreducible curves in $\mathcal{H}$ are rational curves and are solutions of the approximate parametrization problem for every irreducible real curve associated to D_H. In addition, we analyze the dimension of $\mathcal{H}$ and we show how to compute a general parametrization of a generic element in $\mathcal{H}$. So, we develop the theoretical frame that allows to describe a linear projective space containing solutions for the approximate parametrization problem. If the degree of the curve is n we have proved that there always exist monomial rational divisors such that their linear spaces have dimension n. For $n \leq 3$ all possible rational divisors are monomial. For $n = 4$, besides monomial divisors, one can easily produce 1-dimensional linear systems. For $n \geq 5$, in our experiments, besides monomial divisors, we always found non-monomial divisors providing linear systems of dimension 0 (i.e. one curve in each different linear system). Furthermore, as shown in Example 3.12, one might work in the direction of increasing the dimension for particular divisors. As a consequence, we prove that a Hausdorff curve of degree n can always be parametrized approximately. Moreover, the associated projective linear space of solutions can be taken of dimension n.

As a consequence of the results of this paper, one can consider new future directions of research to approach the approximate parametrization problem. For instance, one can work in the direction of analyzing and deciding how to choose the rational divisor D_S to be added to D_H. This would imply a new approximate parametrization algorithm that would simplify some of the required hypotheses in [22], and would have the property of controlling some features of the output parametrization as the length of the coefficients. On the other hand, the potential extension of these ideas to space curves or to surfaces is a natural research extension.

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